THE
ANALYTICAL GEOMETRY
OF
HYPER-SPACES

PART II

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OF
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PART II

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PREFACE

The following pages contain certain interesting results in the Geometry of Hyper-spaces, entirely a modern branch of Mathematics, which is now recognised as an indispensable part of that science with extensive applications in Mathematical Physics. In preparing this volume I have always adopted analytic methods of proof, supplemented by geometrical conceptions where simplification could be gained thereby. The subject covers too extensive a field to be adequately dealt with in its various aspects in a small work like this ; consequently, in the choice of subject-matter I have selected only those which could be treated on the most elementary principles without a knowledge of higher Mathematics. I have carefully avoided the discussions of of geometrical figures which, in higher spaces, are no doubt very complicated and difficult of conception.

In Chapter I, the most prominent properties of isocline planes have been discussed.

In Chapter II, Motion in a space of four dimensions has been studied, but little attempt has been made to extend the ideas to spaces of dimensions beyond the fourth.

In Chapter III, Complexes in n-dimensions have been discussed, giving, in some cases, geometrical interpretations to the equations defining the Continuum. The name " paralleloschem " has been adopted after Schläfli.

In Chapter IV, properties of hyper-surfaces have been discussed at some considerable length. No writer in hyper-geometry has as yet, to my knowledge, attempted a systematic study of the subject ; my attention has however been chiefly confined to a systematic development of the subject on the most elementary principles, although a thorough discussion has to be postponed for want of space.

In Chapter V, only an introduction of the differential methods in the geometry of hyper-spaces has been attempted.

In a modern science like this, great confusion arises with regard to nomenclature. A considerable number of new terms have therefore been introduced, formed, as far as practicable, in accordance with recognised principles. I am indebted to the works of Veronese, Bertini, Schläfli, Manning, Cole, etc., for substantial help in preparing this volume and my obligations to these authors are perhaps greater than I am aware of.

In conclusion, once more I must acknowledge myself in the highest degree indebted to the Hon'ble Sir Asutosh Mookerjee, Vice-Chancellor and President of the Council of Post-Graduate Teaching in Arts, University of Calcutta, for his extreme kindness in often encouraging me to prosecute research in the field of hyper-geometry. It was his most inspiring advice which prompted me to take to the subject and continue the investigation. My best thanks are due to Mr. A. C. Ghatak, B.A., Superintendent, and the Staff of the Calcutta University Press, who took a personal interest in bringing out the book in the least possible time.

UNIVERSITY OF CALCUTTA,
March, 1922.

S. M. GANGULI.

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ANALYTICAL GEOMETRY

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[PART II]

CHAPTER I—Isocline Planes.

1. Angles between two planes :—

We have already defined (Part, I §. 23) *minimum* angles between any two planes and obtained analytical expressions for them. We shall now show that they really define the inclination of one plane to the other.

$$\text{Let } \frac{L_{rs}}{\sin\theta} \text{ and } \frac{L'_{rs}}{\sin\theta'}, \left[\begin{matrix} r \\ s \end{matrix} = 1, 2, 3, 4\right]$$

be the direction-cosines* of the two given planes, both passing through the origin.

Then, by analogy to the formula in our ordinary Geometry of three dimensions, we may assume that some function or functions of the angle or angles between the two planes (which we denote at present by $\cos\Omega$) are given by

$$\cos\Omega = \frac{\Sigma L_{rs}\, L'_{rs}}{\sin\theta\sin\theta'},$$

---

* See the author's paper "On the angle-concept in n-dimensional Geometry"—§ 5. Bulletin of the Calcutta Math. Soc., Vol. IX, No. I.

or $\sin\theta . \sin\theta' \cos\Omega = \Sigma\, \mathrm{L}rs\, \mathrm{L}'rs \left[{r \atop s} = 1, 2, 3, 4\right]$

$$= \Sigma \begin{vmatrix} l_r & l_s \\ m_r & m_s \end{vmatrix} \begin{vmatrix} l'_r & l'_s \\ m'_r & m'_s \end{vmatrix}$$

$$= \begin{vmatrix} l_1 l'_1 + l_2 l'_2 + \ldots. & l_1 m_1' + l_2 m'_2 + \ldots \\ l'_1 m_1 + l'_2 m_2 + \ldots, & m_1 m'_1 + m_2 m'_2 + \ldots \end{vmatrix}$$

$$= \begin{vmatrix} \cos\widehat{ll'} & \cos\widehat{lm'} \\ \cos\widehat{l'm} & \cos\widehat{mm'} \end{vmatrix}$$

$$= \begin{vmatrix} (ll') & (lm') \\ (l'm) & (mm') \end{vmatrix}$$

where $\cos\widehat{ll'} \equiv ll')$ and $\sin\widehat{ll'} \equiv [ll']$.

$$\therefore [lm]\,[l'm'] \cos\Omega = (ll')\,(mm') - (lm')\,(l'm) \quad \ldots \quad (1)$$

To prove that this formula must always hold, we take any other line (L) in the plane (l, m), whose direction-cosines may be taken as—

$$\lambda l_i + \mu m_i, \ [i = 1, 2, 3, 4]$$

where λ and μ are multipliers.

Let this line L make angles ϕ and ψ with the lines (l) and (m) respectively, so that $\theta = \phi + \psi$.

Let Ω_1 be the "Hyper-angle"* between the planes (L, l) and (l', m'), and Ω_2 be that between (L, m) and (l', m'),

---

* We call this Hyper-angle because we do not know the nature of Ω_1 as yet, whether it is a plane or a solid angle, or an angle of any other kind.

Now, we have $L_i = \lambda l_i + \mu m_i$

and $$1 = \Sigma L_i^2 = \lambda^2 + \mu^2 + 2\lambda\mu \cos(\phi+\psi) \quad \ldots \quad (2)$$

If the formula (1) is to be always true, we must have

$$\sin\phi.\ \sin\theta' \cos\Omega_1 = \Sigma \begin{vmatrix} \lambda l_r + \mu m_r & \lambda l_s + \mu m_s \\ l_r & l_s \end{vmatrix} \begin{vmatrix} l'_r & l'_s \\ m'_r & m'_s \end{vmatrix}$$

$$= \mu \Sigma \begin{vmatrix} l_r & l_s \\ m_r & m_s \end{vmatrix} \begin{vmatrix} l'_r & l'_s \\ m'_r & m'_s \end{vmatrix}$$

$$= \mu \sin\theta \sin\theta' \cos\Omega, \text{ by } (1)$$

$$\therefore\ \sin\phi \cos\Omega_1 = \mu \sin(\phi+\psi) \cos\Omega. \quad \ldots \quad (3)$$

Similarly, by considering the planes (L, m) and (l', m') we have

$$\sin\psi \cos\Omega_2 = \lambda \sin(\phi+\psi) \cos\Omega. \quad \ldots \quad (4)$$

If the formula (1) is to be always true, conditions (2), (3), (4), must simultaneously hold and we should have

$$\Omega = \Omega_1 = \Omega_2.$$

Now, $$\cos\phi = \widehat{lL} = \Sigma l_i(\lambda l_i + \mu m_i)$$

$$= \lambda \Sigma l_i^2 + \mu \Sigma l_i m_i$$

$$= \lambda + \mu(lm)$$

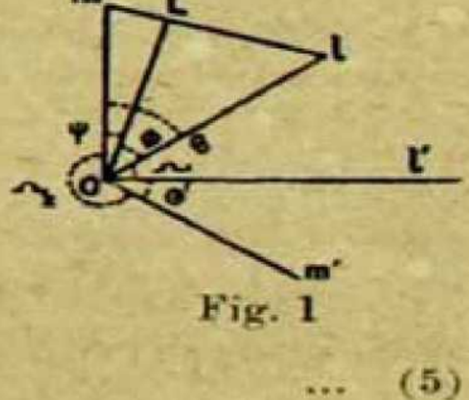

Fig. 1

$$= \lambda + \mu \cos(\phi+\psi) \quad \ldots \quad (5)$$

Similarly, $$\cos\psi = \cos\widehat{Lm} = \mu + \lambda\cos(\phi+\psi) \quad \ldots \quad (6)$$

$$\therefore\ \sin^2\phi = 1 - \cos^2\phi = 1 - \{\lambda + \mu\cos(\phi+\psi)\}^2$$

$$= \mu^2 \sin^2(\phi+\psi), \text{ by } (2)$$

and $$\sin^2 \psi = 1 - \cos^2 \psi = \lambda^2 \sin (\phi + \psi).$$

$$\therefore \frac{\lambda}{\sin \psi} = \frac{\mu}{\sin \phi} = \frac{1}{\sin (\phi + \psi)}.$$

Substituting these values of λ, μ in (3) and (4), we find

$$\cos \Omega_1 = \cos \Omega_2 = \cos \Omega.$$

and therefore, $$\Omega_1 = \Omega_2 = \Omega.$$

and the expression (1) always holds.

Since the expression (1) involves only the functions of angles between the four lines determining the planes, it is independent of the orientation of the system of axes and depends only on the initial lines.

Putting $$[lm/l'm'] \equiv \Sigma \begin{vmatrix} l_r & l_s \\ m_r & m_s \end{vmatrix} \begin{vmatrix} l'_r & l'_s \\ m'_r & m'_s \end{vmatrix}$$

the expression (1) may be written as—

$$\cos \Omega . \sin \theta \sin \theta' = [lm/l'm'].$$

$$\text{or, } \cos^2 \Omega = [lm/l'm']^2 / [lm]^2 [l'm'].^2$$

So far we do not know what Ω or $\cos \Omega$ means. But we have seen that if θ_1 and θ_2 be the minimum angles between two lines, one in each plane,

$$\cos^2 \theta_1 . \cos^2 \theta_2 = [lm/l'm']^2 / [lm]^2 \, [l'm'].^2 \, *$$

Thus we find that $$\cos^2 \Omega \equiv \cos^2 \theta_1 . \cos^2 \theta_2 .$$

i.e., what we have called $\cos \Omega$ is nothing but the product of the cosines of the two minimum angles between the planes.

---

* Vide—Analytical Geometry of Hyper-spaces, Part I, §. 23.

Hence, we may define the magnitude Ω as the "*Hyper angle*" between two planes, when it satisfies the relation

$$\cos^2\Omega=\cos^2\theta_1.\cos^2\theta_2.$$

2. We have seen that the minimum angles between two planes (l, m) and (p, q) are given by the equation :—

$$\cos^4\theta[lm]^2[pq]^2+\{[lmpq]^2-[lm]^2[pq]^2-[lm/pq]^2\}\cos^2\theta+[lm/pq]^2=0.$$

Put

$$[lm]^2[pq]^2\equiv a^2$$

$$[lmpq]^2\equiv b^2$$

$$[lm/pq]^2\equiv c^2.$$

Then the equation becomes—

$$a^2\cos^4\theta+(b^2-a^2-c^2)\cos^2\theta+c^2=0.$$

This is a quadratic in $\cos^2\theta$.

$$\therefore\quad \cos^2\theta=\frac{(a^2+c^2-b^2)\pm\sqrt{(b^2-a^2-c^2)^2-4a^2c^2}}{2a^2}$$

$$=\frac{(a^2+c^2-b^2)\pm\sqrt{a^4+b^4+c^4-2a^2b^2-2b^2c^2-2a^2c^2}}{2a^2}$$

The expression under the radical sign breaks up into factors—

$$-(a+b+c)(b+c-a)(c+a-b)(a+b-c)\quad\ldots\quad(1)$$

$\therefore$ The two roots of the above equation will be equal when any of the factors in (1) vanishes.

The expression under the radical may also be written in the equivalent form

$$\triangle\equiv\begin{vmatrix}0&1&1&1\\1&0&c^2&b^2\\1&c^2&0&a^2\\1&b^2&a^2&0\end{vmatrix}=\begin{vmatrix}0&a&b&c\\a&0&c&b\\b&c&0&a\\c&b&a&0\end{vmatrix}$$

Hence, the condition for the equality of the roots may be written as $\triangle=0$, or

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & [lm/pq]^2 & [lmpq]^2 \\ 1 & [lmpq]^2 & 0 & [lm].^2[pq]^2 \\ 1 & [lmpq]^2 & [lm].^2[pq]^2 & 0 \end{vmatrix}=0$$

When this condition is satisfied the two roots are equal, *i.e.*, $\cos\theta_1=\cos\theta_2$. $\therefore\ \theta_1=\theta_2$ or $-\theta_2$

In this case the two planes are said to be *isocline* to each other.

Definition :—Two planes are said to be *isocline* to each other, when the two minimum angles between them are equal in magnitude. The two planes are called "*isoclines.*"*

3. If two half-lines in the plane α make equal angles with another plane β, then the half-line bisecting the angle between them and the half-line bisecting the angle between their projections upon β will lie in one of the common perpendicular planes of α and β, and the angle between them is equal to the equal angles between the two planes. †

Let l, m be the two half-lines in α and p, q their projections upon β, all passing through a common point.

Let λ and μ be the half-lines bisecting the angles between (l, m) and (p, q) respectively.

$$\therefore \lambda_i=\frac{l_i+m_i}{2}$$

$$\mu_i=\frac{p_i+q_i}{2} \qquad [i=1, 2, 3, 4]$$

---

* *Cf.* § 24, Cor. 3, Ana. Geo., Part I.

† This theorem has been used by Veronese in finding perpendicular planes—Grundzüge der Geometrie, &c. § 150.

and $$1=\Sigma\lambda_1^2=\tfrac{1}{4}[\Sigma l_1^2+\Sigma m_1^2+2(lm)]$$

$$=\tfrac{1}{2}\{1+(lm)\} \qquad \dots \quad (1)$$

Similarly, $$1=\tfrac{1}{2}\{1+(pq)\} \qquad \dots \quad (2)$$

If ϕ be the angle between λ and μ, we have—

$$\cos\phi=\Sigma\lambda_1\mu_1=\tfrac{1}{4}\{\Sigma l_1 p_1+\Sigma m_1 p_1+\Sigma l_1 q_1+\Sigma m_1 q_1\}$$

$$=\tfrac{1}{4}\{(lp)+(mp)+(lq)+(mq)\}.$$

But $$(lp)=(mq)=\cos\theta \text{ (say)}$$

$$\therefore \cos\phi=\tfrac{1}{4}\{2\cos\theta+(mp)+(lq)\}$$

Now, m, p, q determine a three-space and $\angle mqp$ is a right dihedral angle.

$\therefore$ We have, by Spherical Trigonometry,

$$(mp)=(mq)\,(pq)=\cos\theta\,(pq)$$

and $$(lq)=(lp)\,(pq)=\cos\theta\,(pq)$$

$$\therefore \cos\phi=\tfrac{1}{4}\,\{2\cos\theta+2\cos\theta\,(pq)\}=\tfrac{1}{2}\{1+(pq)\}\cos\theta$$

$$=\cos\theta \text{ [by (2)]}$$

Again, $$(\lambda p)=(\lambda l)\,(lp)=\cos\theta\,(\lambda l)$$

$$(\lambda q)=(\lambda m)\,(mq)=(\lambda l)\,(lp)=\cos\theta(\lambda l)$$

$$\therefore (\lambda p)=(\lambda q)$$

But $$(\lambda p)=(\lambda\mu)(\mu p)+[\lambda\mu][\mu p]\cos<\lambda\mu p^{*} \qquad \dots \quad (3)$$

and $$(\lambda q)=(\lambda\mu)\,(\mu q)+[\lambda\mu]\,[\mu q]\cos\angle\lambda\mu p$$

$$=(\lambda\mu)(\mu p)+[\lambda\mu][\mu p]\cos\angle\lambda\mu q \qquad \dots \quad (4)$$

$\therefore$ From (3) and (4) we have—

$$\text{Cos}\angle\lambda\mu p=\text{Cos}\angle\lambda\mu q$$

i.e., the dihedral angle $\lambda\mu p$ = the dihedral angle $\lambda\mu q$;
i.e., the plane of (λ,μ) is perpendicular to the plane of (p, q)
i.e., to β.

---

* Tod Hunter, Sph. Tri., § 42.

Similarly, it may be shewn that the plane (λ, μ) is perpendicular to the plane α.

$\therefore$ The half-lines λ and μ determine a common perpendicular plane of the two isocline planes α and β; and the angle between them is equal to the "*isoclinal angle*" between the planes.

Cor. :—If more than two pairs of opposite half-lines in one of two planes make any given angle with the other plane, the two planes are *isocline* and the given angle is called the "*isoclinal angle*" between the planes.

4. The generalised form of the above theorem is the following :—

If λ and μ be taken one in each plane, dividing the angles between the "*minimal lines*" in the same ratio, then the plane of (λ, μ) is perpendicular to both α and β and the angle between λ and μ is equal to the isoclinal angle.

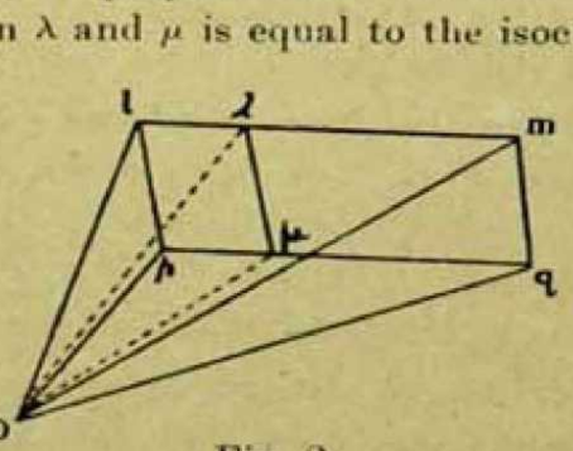

Fig. 2

Let
$$\left.\begin{aligned}\lambda_i &= \mathrm{A}l_i + \mathrm{B}m_i\\ \mu_i &= \mathrm{A}p_i + \mathrm{B}q_i\end{aligned}\right\} \quad [i=1, 2, 3, 4]$$

and
$$1 = \Sigma\lambda_i^2 = \mathrm{A}^2 + \mathrm{B}^2 + 2\ \mathrm{AB}(lm)$$

and
$$1 = \Sigma\mu_i^2 = \mathrm{A}^2 + \mathrm{B}^2 + 2\ \mathrm{AB}(pq)$$

$$\begin{aligned}\cos\phi &= \Sigma\lambda_i\mu_i = \Sigma(\mathrm{A}l_i + \mathrm{B}m_i)(\mathrm{A}p_i + \mathrm{P}q_i)\\ &= \mathrm{A}^2\ (lp) + \mathrm{B}^2(mq) + \mathrm{AB}(lq) + \mathrm{AB}(mp)\\ &= \mathrm{A}^2\ \cos\theta + \mathrm{B}^2\ \cos\theta + \mathrm{AB}\ \cos\theta(pq) + \mathrm{AB}\ \cos\theta(pq)\\ &= \cos\theta\{\mathrm{A}^2 + \mathrm{B}^2 + 2\mathrm{AB}(pq)] = \cos\theta.\end{aligned}$$

Again,

$$(\lambda q) = \Sigma(Al_i + Bm_i)q_i = A(lq) + B(mq)$$
$$= A\cos\theta(pq) + B\cos\theta$$
$$= \cos\theta\{A(pq) + B\} \qquad \dots \quad (1)$$

$$(\lambda p) = \Sigma(Al_i + Bm_i)p_i = A(lp) + B(mp)$$
$$= A\cos\theta + B\cos\theta(pq)$$
$$= \cos\theta\{A + B(pq)\} \qquad \dots \quad (2)$$

Also, $$(\lambda q) = (\lambda\mu)(\mu q) + [\lambda\mu][\mu q]\cos\angle\lambda\mu q \qquad \dots \quad (3)$$
$$= \cos\theta\{A(pq) + B)\}[\lambda\mu][\mu q]\cos\angle\lambda\mu q.$$

From (1), $$[\lambda\mu][\mu q]\cos\angle\lambda\mu q = 0$$

i.e., $$\cos\angle\lambda\mu q = 0$$

$\therefore$ $\angle\lambda\mu q$ or the dihedral angle $\lambda\mu q$ is a right angle. Similarly, it may be shewn that $\lambda\mu p$ is a right angle, *i.e.*, the plane of (λ, μ) is perpendicular to β. Similarly, it is perpendicular to α.

Cor. :—Hence it follows that any plane meeting two isoclines α and β in two lines which divide the angles between the minimal lines in each in the same ratio is a common perpendicular plane to α and β, and the two planes cut out on it an angle equal to the isoclinal angle.

5. If two planes cut out equal angles on a pair of common perpendicular planes they have an infinite number of common perpendicular planes, on which they cut out equal angles. Unless they are absolutely perpendicular planes, any two of these common perpendicular planes cut out equal angles on the two planes.

Let two planes $\alpha(l, m)$ and β (p, q) have a pair of common perpendicular planes (l, p) and (m, q), on which they cut out the angles $\phi = \widehat{lp}$ and $\phi' = \widehat{mq}$. Consequently l, m and p, q are the minimal lines. (Fig. 2.)

It is known that angle $\widehat{lm}$ and $\widehat{pq}$ are right angles.

In $\widehat{lm}$ and $\widehat{pq}$, lay off equal angles $\widehat{l\lambda}$ and $\widehat{p\mu}$.

Then, evidently the lines λ and μ divide the angles $\widehat{lm}$ and $\widehat{pq}$ in the same ratio and therefore supposing that $\phi=\phi'$, by the corollary of § 4, the plane of (λ, μ) is a common perpendicular plane to both α and β, and the angle $\widehat{\lambda\mu}$ is equal to the isoclinal angle ϕ.

We can make the angles $\widehat{l\lambda}$ and $\widehat{p\mu}$ equal to any given angle and in all cases the plane of (λ, μ) is common perpendicular to α and β, provided the angles $\widehat{l\lambda}$ and $\widehat{pq}$ are equal.

Thus there are infinite number of common perpendicular planes to two "isoclines."

Again, since the half lines of the common perpendicular planes make equal angles with l and p, any two common perpendicular planes must cut out equal angles on α and β, provided α and β are not absolutely perpendicular; for in that case all lines in α are perpendicular to all lines in β, and any plane is a common perpendicular to both.

6. The converse theorem is also true :—

If two planes, not being absolutely perpendicular, have more than two common perpendicular planes, the acute angles which they cut out on any pair of these common perpendicular planes are equal, and the planes are isocline to each other.*

---

* *Cf.* Manning—Geometry of Four Dimensions, § 68.

In the figure of the last article, let (λ, μ) be another common perpendicular plane to α and β, and let $\widehat{lp}=\phi$ and $\widehat{mq}=\phi'$. It is required to prove that $\widehat{\lambda\mu}=\phi=\phi'$.

Suppose we had $\widehat{\lambda\mu}>\phi$ *i.e.* $>\widehat{lp}$.

$$\therefore\ (\lambda\mu)<(lp)$$

But $\quad (\lambda p)=(\lambda\mu)(p\mu)$ and $(l\mu)=(lp)(p\mu)$

$$\therefore (l\mu)>(\lambda p) \qquad \dots \quad (1)$$

Again, $\quad (\lambda p)=(l\lambda)\ (lp)$ and $(l\mu)=(l\lambda)\ (\lambda\mu)$

$$\therefore (\lambda p)>(l\mu) \qquad \dots \quad (2)$$

Inequalities (1) and (2) lead to contradiction.

$$\therefore \text{ We must have } (\lambda\mu)=(lp)=\cos\phi.$$

$$\textit{i.e.},\ \widehat{\lambda\mu}=\phi=\widehat{lp}.$$

In the same way it can be proved that $\widehat{\lambda\mu}=\widehat{mq}=\phi'$.

$$\therefore\ \widehat{\lambda\mu}=\phi=\phi'\,;$$

i.e., the planes α and β are "isoclines."

7. If (l, m) and (p, q) are the "minimal lines" of two planes α and β, and if two half-lines λ and μ be taken, one in each of the planes, dividing the angles between the minimal lines in the same ratio, the plane of (λ, μ) is equally inclined to α and β. The angle cut out on this plane by α and β lies between θ_1 and θ_2, the minimum angles between α and β. Also, the plane of the half lines bisecting the angles θ_1 and θ_2 intersects the plane of (λ, μ) orthogonally and bisects the angle between λ and μ.

Let k and h be the lines bisecting the angles θ_1 and θ_2 and let the plane of (k, h) intersect that of (λ, μ) in the line T.

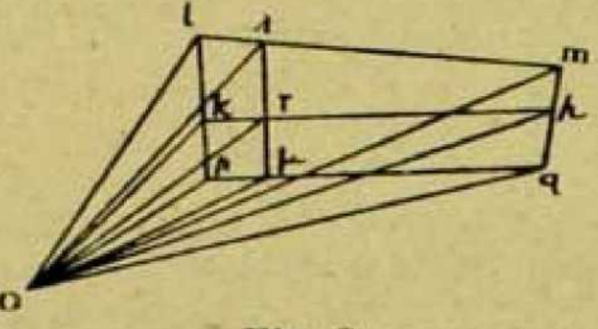

Fig. 3

Also, let
$$\left.\begin{aligned} \lambda_i &= \mathrm{A}l_i + \mathrm{B}m_i \\ \mu_i &= \mathrm{A}p_i + \mathrm{B}q_i \\ k_i &= \frac{l_i + p_i}{2} \\ h_i &= \frac{m_i + q_i}{2} \end{aligned}\right\} [i = 1, 2, 3, 4]$$

We have,
$$(l\mu) = (lp)\,(p\mu) - \cos\theta_1\,(\mu p)$$
$$= \cos\theta_1\,(\lambda l) = (\lambda p)$$

Now,
$$(l\mu) = (l\lambda)\,(\lambda\mu) + [\lambda l]\,[\lambda\mu]\cos\angle l\lambda\mu.$$
$$= (\lambda p)$$

Also
$$(\lambda p) = (p\mu)\,(\lambda\mu) + [p\mu]\,[\lambda\mu]\cos\angle p\mu\lambda$$
$$= (l\lambda)\,(\lambda\mu) + [l\lambda]\,[\lambda\mu]\cos\angle p\mu\lambda$$

$$\therefore \cos\angle l\lambda\mu = \cos\angle p\mu\lambda \text{ } i.e., \angle l\lambda\mu = \angle p\mu\lambda;$$

i.e., the dihedral angle $l\lambda\mu$ = dihedral angle $p\mu\lambda$.

Similarly, $\angle m\lambda\mu = \angle q\mu\lambda$

Let $\widehat{\lambda\mu} = \phi.$

Then, if $\phi > \theta_1, \phi < \theta_2;$

For, $$\cos\angle l\lambda\mu = -\cos\angle m\lambda\mu = \frac{(l\mu)-(l\lambda)(\lambda\mu)}{[l\lambda][\lambda\mu]}$$

$$= \frac{(lp)(p\mu)-(l\lambda)(\lambda\mu)}{[l\lambda][\lambda\mu]}$$

But $$\cos\angle m\lambda\mu = \frac{(\mu m)-(\lambda m)(\lambda\mu)}{[\lambda m][\lambda\mu]}$$

$$= \frac{(mq)(\mu q)-(\lambda m)(\lambda\mu)}{[\lambda m][\lambda\mu]}$$

$$= \frac{\cos\theta_2[p\mu]-[l\lambda](\lambda\mu)}{(l\lambda)[\lambda\mu]}.$$

$$\therefore \frac{\cos\theta_1(p\mu)-(l\lambda)(\lambda\mu)}{[l\lambda][\lambda\mu]} = -\frac{\cos\theta_2[p\mu]-[l\lambda](\lambda\mu)}{(l\lambda)[\lambda\mu]}$$

or $$\cos\theta_1(p\mu)(l\lambda)-(l\lambda)^2(\lambda\mu)$$

$$= -\cos\theta_2[p\mu][l\lambda]+[l\lambda]^2(\lambda\mu)$$

or $$(\lambda\mu)\{(l\lambda)^2+[l\lambda]^2\} = \cos\theta_1(p\mu)(l\lambda)+\cos\theta_2[p\mu][l\lambda]$$

$$= \cos\theta_1(l\lambda)^2+\cos\theta_2[l\lambda]^2$$

But $$(l\lambda)^2+[l\lambda]^2=1$$

$$\therefore (\lambda\mu) = \cos\theta_1(l\lambda)^2+\cos\theta_2[l\lambda]^2$$

$$> (\lambda\mu)(l\lambda)^2+\cos\theta_2[l\lambda]^2$$

for, $$\cos\phi = (\lambda\mu) < \cos\theta_1$$

$$\therefore (\lambda\mu)\{1-(l\lambda)^2\} > \cos\theta_2[l\lambda]^2$$

i.e., $$(\lambda\mu)[l\lambda]^2 > \cos\theta_2[l\lambda]^2$$

or $$(\lambda\mu) > \cos\theta_2 \text{ i.e., } \cos\phi > \cos\theta_2$$

$$\therefore \phi < \theta_2.$$

That is, if ϕ is greater than θ_1, it is less than θ_2, *i.e.*, ϕ lies between θ_1 and θ_2.

The angle (ψ) between the planes of (λ, μ) and (k, h) is given by—

$$\cos^2\psi = [\lambda\mu/kh]^2/[\lambda\mu]^2[kh]^2$$

The numerator is the square of the determinant

$$\begin{vmatrix} (\lambda k) & (\lambda h) \\ (\mu k) & (\mu h) \end{vmatrix} \quad i.e., \text{ of } (\lambda k)\ (\mu h) - (\lambda h)\ (\mu k)$$

$$\text{or, of } \left\{ \Sigma(Al_i + Bm_i)\left(\frac{l_i + p_i}{2}\right)\right\} \left\{ \Sigma(Ap_i + Bq_i)\left(\frac{m_i + q_i}{2}\right)\right\}$$

$$- \left\{ \Sigma(Al_i + Bm_i)\left(\frac{m_i + q_i}{2}\right)\right\} \left\{ \Sigma(Ap_i + Bq_i)\left(\frac{l_i + p_i}{2}\right)\right\}$$

$$i.e., \text{ of } \frac{1}{4}\Big[A^2\{(mp) + (lp)(mp) - (lp)(lq) - (lq)\}$$

$$+ B^2\{(mp)(mq) + (mp) - (lq) - (lq)(mq)\}$$

$$+ AB\{(mp)^2 - (lq)^2\}\Big]$$

But $\qquad (mp) = 0 \qquad$ and $\qquad (lq) = 0.$

$\therefore$ The numerator becomes zero ; *i.e.*, $\cos\psi = 0$

$$\therefore \ \psi = \frac{\pi}{2}.$$

$\therefore$ The planes of (λ, μ) and (k, h) are mutually perpendicular. Also, the direction-cosines of T, the line of intersection of the planes (λ, μ) and (k, h) are obtained as follows :—

If T is the line which divides the angle $\widehat{kh}$ in the ratio of A : [illegible], its direction-cosines are proportional to—

$$T_i = A\,\frac{l_i + p_i}{2} + B\,\frac{m_i + q_i}{2},$$

$$= \tfrac{1}{2}\{(Al_i + B\,m_i) + (Ap_i + B\,q_i)\}$$

$$= \frac{\lambda_i + \mu_i}{2}, \qquad [i = 1, 2, 3, 4].$$

This shows that the line T bisects the angle between λ and μ.*

8. Conjugate series of isocline planes:—Let the planes α and β be two isoclines and let the plane α' be absolutely perpendicular to α, so that the four terminal half-lines l, m, l', m' form a rectangular system.

The two planes α and β have an infinite number of common perpendicular planes (§ 5) on which they cut out equal angles and any two of these common perpendicular planes cut out equal angles on α and β.

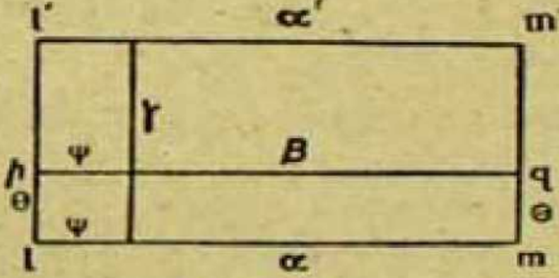

Fig. 4

Thus, if in the planes α and β we lay off an angle ψ from l and p in the same sense, the half-lines of these angles will form an angle θ, and will determine a plane γ perpendicular to both α and β. This plane γ is perpendicular to both the planes α and α', and consequently is one of the common perpendicular planes of β and α'. The plane γ makes with l' an angle ψ and can therefore be determined by laying off angles ψ from l and l' and the construction can be performed without any reference to the plane β. Thus this plane γ is a common perpendicular plane to all the planes of the isocline series β, obtained by giving different values to θ. Similarly, by giving different values to ψ we have an infinite number of planes γ perpendicular to all the planes of the series β.

---

* The angle $\widehat{\lambda\mu}$ has been called an isoclinal angle of the planes α and β, and the angles θ_1 and θ_2 are its maximum and minimum values. (*Vide*—Trans. of the American Mathematical Society, Vol. II, 1901, page 109, paper by I. Stringham.) Stringham says that "Two planes may be said to be mutually isoclinal, when their isoclinal angle is constant," and he calls the two planes "isoclines." We have called the constant angle "isoclinal angle between the two planes."

Thus we have obtained two series of isocline planes, β series, and γ series, each plane of the one series being perpendicular to all planes of the other series. These two series of isoclines are called "*conjugate series of isocline planes.*"

9. Senses of isoclinism * :—In the above construction we have taken $\theta=\theta'$. But the two planes a and β will still be isocline if we make $\theta=-\theta'$. These two cases are quite distinct, and in the latter case we measure $\theta'=-\theta$ in the plane (m, m') on the side of m remote from q and if q' be the other arm of angle, we have $\widehat{lp}=\theta$ and $\widehat{mq'}=-\theta$, and the plane determined by (p, q') is isocline to a, but in a sense opposite that in which the plane (p, q) is isocline. Thus both the planes (p, q) and and (p, q') are isocline to a but in opposite senses. If we lay off equal angles on (l, l') and (m, m') on the same side as p, q; and p'', q'' be the other arms of these angles, then the planes (p, q) and (p'', q'') are isocline to a in the same sense.

When two planes are isocline in opposite senses, one is said to be *positively isocline* and the other *negatively isocline.*

10. We may here quote (without proof) two very important theorems regarding isoclines :

Theorem A :—If a plane is isocline to another, the latter will be isocline to the former in the same sense.

---

* I. Stringham—Transactions of the American Mathematical Society, Vol. II, 1901.

Theorem B :—Two conjugate series of isoclines are isocline in opposite senses.

Cor. :—Two absolutely perpendicular planes are isocline in both senses, but only in one sense when we distinguish in each a particular direction of rotation.

11. Given a half-line (p), not in a given plane a, nor perpendicular to it. Determine the number of planes which can be drawn through (p), isocline to a and also their senses of isoclinism.

Let (l) be the projection of p on the plane a. Then the plane of (p, l) is perpendicular to a. Let (m, q) be the plane absolutely perpendicular to the plane (p, l) and let m be the line of intersection of a with (m, q). In the plane (m, q) lay off angles $\widehat{mq}$ and $\widehat{mq'}$ (equal to $\widehat{pl}$), measured from m in opposite directions. The terminal half-lines of these angles are q and $q.'$ Then p and q determine a plane which is isocline to a in one sense, but p and q' determine another plane through p isocline to a in opposite sense. Thus we see that there are two planes which can be passed through (p) isocline to a in opposite senses.

The construction is this: From m lay off (in the plane (q, m) angles θ and $-\theta$, where $\theta = \widehat{pl}$.

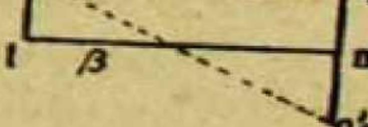

Fig. 5

In (p, q), p and q are the minimal lines, while in (p, q'), p and q' are the minimal lines.

12. The two common perpendicular planes of a given plane a and two other isoclines in the same sense, drawn through a line of α, are inclined to each other at an angle, constant for all positions of the line.

Let (l, m) be the given plane α and α' (l', m') its absolutely perpendicular plane. Also, let β (p, q) and γ (p', q') be two planes isocline in the same sense to α. It is required to prove that the angle between the common perpendicular planes to the two series which pass through any half-line (l) in α is constant.

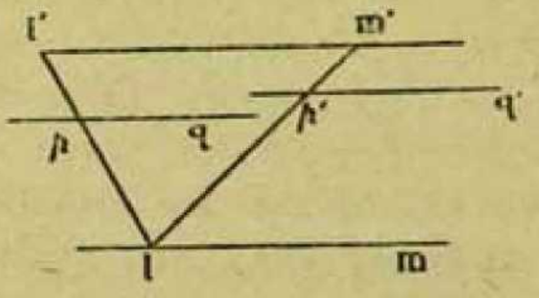

Fig. 6

Let (l, p) be a common perpendicular plane to α and β and (l, p') to α and γ. Let (l, p) and (l, p') meet the plane α' in the half-lines l' and m'.

We have seen that the common perpendicular planes of α and γ are none of them perpendicular to the common perpendicular plane of α and β, except α and α'.

Now we have

$\cos l'lm' = \dfrac{(l'm') - (ll')(lm')}{[ll'][lm']}$, for the half lines l, l', m' lie in a three-space.

But $\widehat{ll'}$ is a right angle, so also is $\widehat{lm'}$ a right angle.

$\therefore \quad (ll') = 0,\ (lm') = 0,\ [ll'] = 1,\ [lm'] = 1$

$$\therefore \quad \cos \widehat{l'lm'} = (l'm') = \cos \widehat{l'm'}$$

i.e., the dihedral angle $\widehat{l'lm'}$ is measured by its opposite face angle $\widehat{l'm'}$.

But the angle $\widehat{l'm'}$ is constant.[*]

$\therefore$ The dihedral angle $\widehat{l'lm'}$ is constant for all positions of the half-line (l) in α.

Note :—If the constant angle vanishes, the series γ is not different from the series β. In that case β and γ belong to the same system of isocline planes with α, and they have a common perpendicular plane. And since the plane α is determined by any two half-lines in it and the above remarks apply equally to the other half-line m in α, we obtain the following theorem as a corollary :—

Corollary :—If a plane α and two isoclines (in the same sense) have a single pair of common perpendicular planes, perpendicular to all three, then all the common perpendicular planes of α and any of them are perpendicular to all three and they all belong to the same series of isoclines.

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[*] We have seen in §8 that the rectangular system of half-lines determining α and α' being rotated give a new system isocline to α and α', *i.e.*, the system γ is obtained from β by rotating the lines in α' through a certain angle. Thus the angle $l'm'$ will be the same for the same planes α and α'

CHAPTER II—Motion in Hyper-space.

13. Before proceeding to the analytical study of the motion in a Hyper-space (of four dimensions), it is advisable and convenient to begin by investigating certain interesting properties which a plane (two-way space) and its absolutely perpendicular plane at any point possess.

A two-way space (a plane in the ordinary space) may be defined in two ways:—

(1) By any two lines in the plane;

(2) By any two three-way spaces containing the plane.

We have already discussed at some length the properties of a plane as determined by two given lines in it and have defined the direction-cosines of the plane and the mutual relations between them.

Now we shall define the plane by two three-way spaces* whose equations are—

$$\begin{cases} l_1 x_1 + l_2 x_2 + l_3 x_3 + l_4 x_4 = 0 \\ l'_1 x_1 + l'_2 x_2 + l'_3 x_3 + l'_4 x_4 = 0 \end{cases}$$

The quantities $l_1, l_2, \ldots ; l_1', l_2 \ldots$ are proportional to the direction-cosines of the normals to the two three-way spaces respectively.

Consider the following determinants—

$$\left\{\begin{matrix} l_1 l'_2 - l_2 l'_1, & l_2 l'_3 - l_3 l'_2, & l_3 l'_4 - l_4 l'_3, \\ l_1 l'_3 - l_3 l'_1, & l_2 l'_4 - l_4 l'_2, & l_1 l'_4 - l_4 l'_1, \end{matrix}\right\}$$

---

* Since we are concerned with the mutual relations between angles, we assume that the plane passes through the origin.

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We have denoted these functions respectively by—

$$L_{12},\quad L_{13},\quad L_{23},\quad L_{24},\quad L_{34},\quad L_{14}.$$

These six quantities are connected by the relation—

$$L_{12}^2+L_{13}^2+L_{23}^2+L_{24}^2+L_{34}^2+L_{14}^2=\sin^2\theta$$

$$=1-(l_1l_1'+l_2l_2'+l_3l_3'+l_4l_4')^2,$$

where θ is the angle between the two normals (1).

Further, by considering the following identically vanishing determinant.

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ l'_1 & l'_2 & l'_3 & l'_4 \\ l_1 & l_2 & l_3 & l_4 \\ l'_1 & l'_2 & l'_3 & l'_4 \end{vmatrix}=0$$

we obtain the relation—

$$L_{12}.L_{34}+L_{13}.L_{42}+L_{14}.L_{23}.=0 \qquad \ldots \quad (2)$$

Thus, the six quantities $L_{12}/\sin\theta$, $L_{13}/\sin\theta$, etc., are connected by two independent conditions (1) and (2). This is as it should be; because a plane through the origin is determined by four constants.

If we put a, b, c, f, g, h for the quantities $L_{12}/\sin\theta$, etc., we obtain the two following relations :—

$$\left.\begin{aligned} a^2+b^2+c^2+f^2+g^2+h^2=1 \\ af+bg+ch=0 \end{aligned}\right\}$$

These quantities a, b, c, f, g, h we have called the "direction-cosines" of the plane determined by the two

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normals (l_1, l_2, l_3, l_4) and (l_1', l_2', l_3', l_4').* If these six quantities are multiplied by a constant (say) k, the six quantities ka, kb,... may be called the "six co-ordinates" of the plane, corresponding to the six Plückerian co-ordinates of a line in three dimensional Geometry.

14. If there are two pairs of three-way spaces such that both the three-ways spaces of one pair are perpendicular to the three-way spaces of the other pair, the plane determined by the first pair is absolutely perpendicular to the plane determined by the other pair. These two planes have in general no line common.

Let the two planes be defined by—

$$(1)\qquad \begin{cases} l_1x_1+l_2x_2+l_3x_3+l_4x_4=0 \\ l'_1x_1+l'_2x_2+l'_3x_3+l'_4x_4=0 \end{cases}$$

$$(2)\qquad \begin{cases} m_1x_1+m_2x_2+m_3x_3+m_4x_4=0 \\ m'_1x_1+m'_2x_2+m'_3x_3+m'_4x_4=0 \end{cases}$$

If the two planes are absolutely perpendicular, the three-way spaces (1) are perpendicular to the three-way spaces (2), or in other words, one plane is defined by the normals in (1) and the other by the normals in (2). The direction-cosines of the two absolutely perpendicular planes have simple relations which we proceed to investigate.

The condition of perpendicularity gives us the four following equations :—

$$\sum_{i=1}^{i=4} l_im_i=0\,;\ \sum_{i=1}^{i=4} l_im'_i=0\,;\ \sum_{i=1}^{i=4} l'_im_i=0\,;\ \sum_{i=4}^{i=4} l'_im'_i=0.$$

---

* These have been called by F. N. Cole "the direction-cosines" of the plane determined by the two three-way spaces—American Journal of Mathematics, Vol. XII, 1890. Thus it is seen that what I call the direction-cosines of a plane are those of its absolutely perpendicular plane according to Cole.

From these equations we deduce the following by simple algebraic operations :—

$$\left.\begin{aligned}
&L_{12}m_2+L_{13}m_3+L_{14}m_4=0\\
&L_{12}m'_2+L_{13}m'_3+L_{14}m'_4=0\\
&L_{12}m_1-L_{23}m_3-L_{24}m_4=0\\
&L_{12}m'_1-L_{23}m'_3-L_{24}m'_4=0\\
&L_{13}m_1+L_{23}m_2-L_{34}m_4=0\\
&L_{13}m'_1+L_{23}m'_2-L_{34}m'_4=0\\
&L_{14}m_1+L_{24}m_2+L_{34}m_3=0\\
&L_{14}m'_1+L_{24}m'_2+L_{34}m'_3=0
\end{aligned}\right\} \quad \ldots \quad (3)$$

From these again the following relations may be deduced, in which M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34} denote functions analogous to L_{12}, etc.

$$\left.\begin{aligned}
&L_{12}M_{24}+L_{13}M_{34}=0\\
&L_{12}M_{23}-L_{14}M_{34}=0\\
&L_{13}M_{12}+L_{34}M_{24}=0\\
&L_{13}M_{14}-L_{23}M_{24}=0
\end{aligned}\right\}
\quad
\left.\begin{aligned}
&L_{12}M_{13}+L_{24}M_{34}=0\\
&L_{12}M_{14}-L_{23}M_{34}=0\\
&L_{14}M_{13}-L_{24}M_{23}=0\\
&L_{14}M_{12}-L_{34}M_{23}=0
\end{aligned}\right\}$$

or

$$M_{24}=-\frac{L_{13}}{L_{12}}M_{34}; \qquad M_{13}=-\frac{L_{24}}{L_{12}}M_{34};$$

$$M_{23}=\frac{L_{14}}{L_{12}}M_{34}; \qquad M_{14}=\frac{L_{13}}{L_{12}}M_{34};$$

$$M_{12}=\frac{-L_{34}}{L_{13}}M_{24}; \qquad M_{14}=\frac{L_{23}}{L_{13}}M_{24}, \text{ etc.}$$

i.e., we have the following five relations :—

$$M_{12}=\frac{L_{34}}{L_{12}}M_{34}; \quad M_{13}=\frac{L_{42}}{L_{12}}M_{34}; \quad M_{14}=\frac{L_{23}}{L_{12}}M_{34}.$$

$$M_{23}=\frac{L_{14}}{L_{12}}M_{34}; \quad M_{24}=\frac{L_{31}}{L_{12}}M_{34}.$$

But $\Sigma M_{12}^2 = \sin^2\theta'$ and $\Sigma L_{12}^2 = \sin^2\theta$.

$$\therefore \sin^2\theta' = \frac{L_{42}^2 + L_{23}^2 + L_{14}^2 + L_{31}^2 + L_{34}^2}{L_{12}^2}\, M_{34}^2 + M_{34}^2$$

$$= \frac{(L_{12}^2 + L_{13}^2 + L_{14}^2 + L_{23}^2 + L_{24} + L_{34}^2) M_{34}^2}{L_{12}^2}$$

$$= \frac{\sin^2\theta .\ M_{34}^2}{L_{12}^2}$$

or
$$\frac{L_{12}^2}{\sin^2\theta} = \frac{M_{34}^2}{\sin^2\theta'} \qquad \therefore\ \frac{L_{12}}{\sin\theta} = \frac{\pm M_{34}}{\sin\theta'}.$$

Taking the positive sign we have—

$$\left.\begin{array}{ll}
\dfrac{L_{12}}{\sin\theta} = \dfrac{M_{34}}{\sin\theta'} & \dfrac{L_{23}}{\sin\theta} = \dfrac{M_{14}}{\sin\theta'} \\[2ex]
\dfrac{L_{13}}{\sin\theta} = \dfrac{M_{42}}{\sin\theta'} & \dfrac{L_{24}}{\sin\theta} = \dfrac{M_{31}}{\sin\theta'} \\[2ex]
\dfrac{L_{14}}{\sin\theta} = \dfrac{M_{23}}{\sin\theta'} & \dfrac{L_{34}}{\sin\theta} = \dfrac{M_{12}}{\sin\theta'}
\end{array}\right\} \quad \dots \quad (4)$$

If a, b, c, f, g, h, and a', b', c', f', g', h', be the direction-cosines of the two absolutely perpendicular planes (l, l') and (m, m'), then we obtain the following relations connecting them :—

$$a = f' \qquad f = a'$$
$$b = -g' \text{ and } g = -b'$$
$$c = h' \qquad h = c'.$$

Further, we have—

$$L_{12} \cdot L_{34} + L_{42} \cdot L_{13} + L_{23} \cdot L_{14}$$
$$= M_{12} .\ M_{34} + M_{13} .\ M_{42} .\ M_{23} = 0$$

or, $$af + bg + ch = a'f' + b'g' + c'h' = 0.$$

Thus we see that if a, b, c, f, g, h be the direction-cosines of a plane, then $f, -g, h, a, -b, c$ are the corresponding direction-cosines of its absolutely perpendicular plane.

N.B.—It is to be noticed here that the plane in (1) is determined by its two normals l and l', or by two lines m and m' which lie in it; so also the plane (2) is determined by its two lines l and l', or by two normals m and m'. In fact, the two planes being mutually absolutely perpendicular, we may consider them as determined by the lines (l, l') and (m, m'). Thus follow the results we have already obtained.

15. The condition that the two planes will intersect in a line is obtained by expressing the fact that the four three-way spaces have another common point besides the origin and this gives—

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ l'_1 & l'_2 & l'_3 & l'_4 \\ m_1 & m_2 & m_3 & m_4 \\ m'_1 & m'_2 & m'_3 & m'_4 \end{vmatrix} = 0;$$

or, expanding the determinant in terms of minors of the second order, we obtain at once :—

$$L_{12}.M_{34} + L_{13}.M_{42} + L_{14}.M_{23} + L_{23}.M_{14} + L_{42}.M_{13} + L_{34}.M_{12} = 0.$$

Thus, two planes whose direction-cosines are a, b, c, f, g, h; $a', b', c', \ldots$ will intersect in a line, if

$$af' + bg' + ch' + a'f + b'g + c'h = 0.$$

This corresponds to Plucker's condition for the intersection of two lines.*

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* Salmon's Geometry of Three Dimensions, Vol. I, § 51.

16. Consider the planes α, determined by the lines (l, l'), and β determined by the lines (m, m'). The direction-cosines of these planes are as before represented by L_{12}, L_{13}, etc., and M_{12}, M_{13}, etc. If β intersects the plane α', absolutely perpendicular to α, from what has been said above it follows that—

$$M_{12}.L_{12}+M_{13}.L_{13}+M_{14}.L_{14}+M_{23}.L_{23}$$
$$+M_{24}.L_{24}+M_{34}.L_{34}=0.$$

Hence, if the direction-cosines of the planes α and β are proportional to a, b, c, etc., and a', b', c', etc., respectively, and β intersects α' in a line, we have

$$aa'+bb'+cc'+ff'+gg'+hh'=0. \qquad \ldots \quad (1)$$

Hence, if θ_1 and θ_2 be the minimum angles between α and β, we have $\cos\theta_1.\cos\theta_2=0$, which shows that the planes α and β are *simply perpendicular* to each other.*

This is quite analogous in form to the condition of perpendicularity of two planes or lines in the ordinary space.

17. If a plane α intersects another plane β and its absolutely perpendicular plane β' in a line, we have the two following conditions :—

$$af'+bg'+ch'+a'f+b'g+c'h=0$$

Also, $$aa'+bb'+cc'+ff'+gg'+hh'=0.$$

---

* *Vide*—Ana. Geo., Part I, § 24.

These symmetrical results show that the plane β intersects not only the plane α, but also its absolutely perpendicular plane α'. The relation between the planes α and β is a reciprocal one and the above relations say that α and β lie in the same three-space and are perpendicular to each other. They intersect in a line, as also do their absolutely perpendicular planes, and further each of the planes intersects the plane absolutely perpendicular to the other and *vice-versá*. Hence, any plane intersecting a plane and its absolutely perpendicular plane is a common perpendicular to both.

Note.—Cole has defined two such planes as *simply perpendicular* planes.

18. Rotation in Hyper-space.*

Consider the effect of changing the directions of axes (without changing the origin) in a Hyper-space of four dimensions. Let $\overset{(j)}{l}_i$ $(j=1,\ 2,\ 3,\ 4\ ;\ i=1,\ 2,\ 3,\ 4)$ be the direction-cosines of a set of new rectangular axes, referred to the original system. If $(x'_1,\ x'_2,\ x'_3,\ x'_4)$ be the coordinates of a point referred to the new axes, we have the following formulae of transformation :—

$$\left.\begin{aligned}
x'_1 &= \overset{(1)}{l}_1 x_1 + \overset{(2)}{l}_1 x_2 + \overset{(3)}{l}_1 x_3 + \overset{(4)}{l}_1 x_4 \\
x'_2 &= \overset{(1)}{l}_2 x_1 + \overset{(2)}{l}_2 x_2 + \overset{(3)}{l}_2 x_3 + \overset{(4)}{l}_2 x_4 \\
x'_3 &= \overset{(1)}{l}_3 x_1 + \overset{(2)}{l}_3 x_2 + \overset{(3)}{l}_3 x_3 + \overset{(4)}{l}_3 x_4 \\
x'_4 &= \overset{(1)}{l}_4 x_1 + \overset{(2)}{l}_4 x_2 + \overset{(3)}{l}_4 x_3 + \overset{(4)}{l}_4 x_4
\end{aligned}\right\} \quad \cdots \quad (1)$$

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* F. N. Cole's Paper—Amer. Journal of Math., Vol. XII, 1890.

These sixteen direction-cosines are connected by the ten following equations :—

$$\left.\begin{array}{lll}
\sum\limits_{i=1}^{i=4}(l_i^{(1)})^2=1\,; & & \sum\limits_{i=1}^{i=4}(l_i^{(2)})^2=1\,; \\
\sum\limits_{i=1}^{i=4}(l_i^{(3)})^2=1\,; & & \sum\limits_{i=1}^{i=4}(l_i^{(4)})^2=1\,; \\
\sum\limits_{i=1}^{i=4}l_i^{(1)}l_i^{(2)}=0\,; & \sum\limits_{i=1}^{i=4}l_i^{(1)}l_i^{(3)}=0\,; & \sum\limits_{i=1}^{i=4}l_i^{(1)}l_i^{(4)}=0\,; \\
\sum\limits_{i=1}^{i=4}l_i^{(2)}l_i^{(3)}=0\,; & \sum\limits_{i=1}^{i=4}l_i^{(2)}l_i^{(4)}=0\,; & \sum\limits_{i=1}^{i=4}l_i^{(3)}l_i^{(4)}=0.
\end{array}\right\}\quad(2)$$

If we apply this transformation to the equation of the Hyper-sphere, namely $x_1^2+x_2^2+x_3^2+x_4^2=\mathrm{R}^2$, the equation remains unchanged in form, *i.e.*, it becomes—

$$x'^2_1+x'^2_2+x'^2_3+x'^2_4=\mathrm{R}^2,$$

in virtue of the relations (2).

Now, instead of changing the directions of axes, we might as well consider the Hyper-sphere rotated about the centre (origin) into itself. This enables us to view the subject from a different and more general standpoint, namely,—that of orthogonal transformation.

Consider the following general transformation :—

$$x'_1=a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5$$

$$x'_2=b_1x_1+b_2x_2+b_3x_3+b_4x_4+b_5$$

$$x'_3=c_1x_1+c_2x_2+c_3x_3+c_4x_4+c_5$$

$$x'_4=d_1x_1+d_2x_2+d_3x_3+d_4x_4+d_5$$

If the origin is to remain unchanged, we must have

$$a_5=b_5=c_5=d_5=0$$

If the transformation is orthogonal, the Hyper-sphere transforms into itself and we have

$$\left.\begin{aligned}
a_1^2+b_1^2+c_1^2+d_1^2&=1\\
a_2^2+b_2^2+c_2^2+d_2^2&=1\\
a_3^2+b_3^2+c_3^2+d_3^2&=1\\
a_4^2+b_4^2+c_4^2+d_4^2&=1\\
a_1a_2+b_1b_2+c_1c_2+d_1d_2&=0\\
a_1a_3+b_1b_3+c_1c_3+d_1d_3&=0\\
a_1a_4+b_1b_4+c_1c_4+d_1d_4&=0\\
a_2a_3+b_2b_3+c_2c_3+d_2d_3&=0\\
a_2a_4+b_2b_4+c_2c_4+d_2d_4&=0\\
a_3a_4+b_3b_4+c_3c_4+d_3d_4&=0
\end{aligned}\right\} \quad \cdots \quad (3)$$

Thus it appears that the constants involved in the above scheme are proportional to the direction-cosines used in the previous scheme. Since the sixteen constants are connected by the ten equations of condition (3), it is possible to express them in terms of six independent others, and the rotation about a point consists of ∞^6 different operations.

Prof. Cayley* has given a method of expressing the n^2 co-efficients of orthogonal transformations in terms of $(\frac{1}{2}n.\overline{n-1})$ independent constants. In the present case $n=4$ and the number of independent constants is 6. Call them a, b, c, f, g, h.

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* Cayley—Crelle's Journal, Vol. XXXII.

Then the 16 co-efficients may be expressed as follows :—

$$ka_1=1-\Delta^2+f^2-a^2+g^2-b^2+h^2-c^2$$
$$=1+f^2+g^2+h^2-a^2-b^2-c^2-\Delta^2.$$
$$ka_2=2(a+\Delta f-bh+cg)$$
$$ka_3=2(b+\Delta g-cf+ah)$$
$$ka_4=2(c+\Delta h-ag+bf)$$
$$kb_1=2(-a-\Delta f+cg-bh)$$
$$kb_2=1+f^2+b^2+c^2-a^2-g^2-h^2-\Delta^2$$
$$kb_3=2(h+\Delta c+fg-ab)$$
$$kb_4=2(-g-\Delta b+hf-ac)$$
$$kc_1=2(-b-\Delta g-cf+ah)$$
$$kc_2=2(-h-\Delta c+fg-ab)$$
$$kc_3=1+g^2+c^2+a^2-b^2-h^2-f^2-\Delta^2$$
$$kc_4=2(f+\Delta a+gh-bc)$$
$$kd_1=2(-c-h\Delta+bf-ag)$$
$$kd_2=2(g+\Delta b+fh-ac)$$
$$kd_3=2(-f-\Delta a+gh-bc)$$
$$kd_4=1+h^2+a^2+b^2-c^2-f^2-g^2-\Delta^2$$

where $\Delta\equiv af+bg+ch$, and

$$k=1+a^2+b^2+c^2+f^2+g^2+h^2+\Delta^2.$$

19. By the above transformation the origin remains fixed and the Hyper-sphere is transformed into itself. Let us examine whether there are other points besides the origin which remain unaltered by this transformation. If such points exist, their co-ordinates must satisfy the equations (obtained by putting x_1, x_2, x_3, x_4 respectively for x'_1, x'_2, x'_3, x'_4 in the scheme of transformation), *i.e.*, they must satisfy

$$\left.\begin{aligned}(a_1-1)x_1+a_2x_2+a_3x_3+a_4x_4&=0\\ b_1x_1+(b_2-1)x_2+b_3x_3+b_4x_4&=0\\ c_1x_1+c_2x_2+(c_3-1)x_3+c_4x_4&=0\\ d_1x_1+d_2x_2+d_3x_3+(d_4-1)x_4&=0\end{aligned}\right\}\quad\ldots\quad(1)$$

These equations are satisfied by the values (0, 0, 0, 0) of the variables which of course correspond to the origin. If other points are to remain fixed in the transformation, the equations (1) will be satisfied by values of the variables different from zero, and we shall have

$$\begin{vmatrix} a_1-1 & a_2 & a_3 & a_4\\ b_1 & b_2-1 & b_3 & b_4\\ c_1 & c_2 & c_3-1 & c_4\\ d_1 & d_2 & d_3 & d_4-1\end{vmatrix}=0\quad\ldots\quad(2)$$

If in this determinant we substitute the values of a's b's, etc., and simplify, it reduces to Δ^2/k. Thus the determinant does not identically vanish, and the vanishing of the determinant requires that $\Delta=0$ or $af+bg+ch=0$. Hence $\Delta=0$ is the condition necessary for the existence of points which remain fixed in the transformation.

When the condition $\Delta=0$ is satisfied, the four equations (1) reduce simply to two independent equations :—

$$(3)\left\{\begin{aligned}&(-a^2-b^2-c^2)x_1+(a-bh+cg)x_2+(b-cf+ah)x_3\\&\qquad\qquad+(c-ag+bf)x_4=0\\&(-a+cg-bh)x_1+(-a^2-g^2-h^2)x_2+(h+fg-ab)x_3\\&\qquad\qquad+(-g+hf-ac)x_4=0\end{aligned}\right.$$

Thus the points of the plane determined by (3) remain fixed after transformation and the rotation like this in which a plane remains fixed is called a "simple rotation."

The functions L_{12}, L_{13}, etc., determined by the two normals to this plane are found to be a^2k^2, abk^2, ack^2, afk^2, $-agk^2$, ahk^2, where $k=1+a^2+b^2+c^2+f^2+g^2+h^2$.

Consequently the direction-cosines of the plane are proportional to f, g, h, c, $-b$, a.

Hence we see that the six independent constants used in the transformation are proportional to the direction-cosines of the fixed plane of the rotation.

20. This fixed plane is not only converted into itself but its individual points also remain fixed, and we say that the Hyper-sphere rotates about this fixed plane. This plane is called the "*axis plane*" of the rotation. In fact, all figures in a Hyper-space of four dimensions may be rotated about some plane as the axis-plane.

The plane absolutely perpendicular to the axis plane is converted into itself, but its individual points do not remain fixed. In fact the absolutely perpendicular plane rotates through a certain angle about the point where it meets the axis plane, *i.e.*, the origin. This angle is called the "*angle of rotation*" for the transformation of the Hyper-space.

We have seen that the Hyper-sphere transforms into itself and consequently the circle in which the absolutely perpendicular plane intersects the Hyper-sphere is converted into itself. Thus the two polar circles * are converted into themselves.

We conclude therefore that the rotation of the Hyper-sphere on itself is the same as the rotation of the point-Geometry at its centre. In a simple rotation, a certain great circle—the axis plane of rotation—remains fixed in all of its points; while its polar great circle—called the circle of rotation—rotates or slides on itself.

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* See B. P. Manning—Geometry of Four Dimensions,—§ 121.

21. The angle of rotation :

We know that a certain plane is determined by four conditions. But the direction-cosines of the plane of rotation are connected by the relation $\triangle \equiv af+bg+ch=0$, and are therefore equivalent to five independent constants. Thus, one degree of freedom is left to the plane of rotation and the angle of rotation is a function of a single constant

If we take any line (l_1, l_2, l_3, l_4) in the absolutely perpendicular plane through the origin, this line is rotated through a certain angle θ, which is called the *angle of rotation.* If (l'_1, l'_2, l'_3, l'_4) be the new position of the line, then we have—

$$\cos\theta = l_1 l'_1 + l_2 l'_2 + l_3 l'_3 + l_4 l'_4.$$

Now, we can substitute the values of (l'_1, l'_2, l'_3, l'_4) in this from the scheme of § 18 and thus determine θ.

Thus, we obtain—

$$\begin{aligned}\cos\theta = {} & a_1 l_1^2 + b_2 l_2^2 + c_3 l_3^2 + d_4 l_4^2 \\ & + (a_2+b_1) l_1 l_2 + (a_3+c_1) l_1 l_3 + (a_4+d_1) l_1 l_4 \\ & \quad + (b_3+c_2) l_2 l_3 + (b_4+d_2) l_2 l_4 + (c_4+d_3) l_3 l_4.\end{aligned}$$

$$\begin{aligned}\therefore k\cos\theta = {} & (1+f^2+g^2+h^2-a^2-b^2-c^2) l_1^2 \\ & + (1+f^2+b^2+c^2-a^2-g^2-h^2) l_2^2 \\ & + (1+g^2+c^2+a^2-b^2-f^2-h^2) l_3^2 \\ & + (1+h^2+a^2+b^2-c^2-f^2-g^2) l_4^2 \\ & + 4(cg-bh) l_1 l_2 + 4(ah-cf) l_1 l_3 + 4(bf-ag) l_1 l_4 \\ & + 4(fg-ab) l_2 l_3 + 4(hf-ac) l_2 l_4 + 4(gh-bc) l_3 l_4.\end{aligned}$$

where $$k=1+f^2+g^2+h^2+a^2+b^2+c^2,$$

and $$\Delta=af+bg+ch=0$$

$$\begin{aligned}\therefore k\cos\theta=k&-2(a^2+b^2+c^2)l_1^2-2(a^2+g^2+h^2)l_2^2\\&-2(b^2+f^2+h^2)l_3^2-2(c^2+f^2+g^2)l_4^2\\&+4(cg-bh)l_1l_2+4(ah-cf)l_1l_3+4(bf-ag)l_1l_4\\&+4(fg-ab)l_2l_3+4(hf-ac)l_2l_4+4(gh-bc)l_3l_4.\qquad\ldots\quad\ldots\quad\ldots\quad(1)\end{aligned}$$

If we take a normal to the axis-plane for the line (l_1, l_2, l_3, l_4), the above expression will give the value of $\cos\theta$.

We may take the normal to a three-way space (3) of § 19 for the normal to the plane :

Then, $l_1 : l_2 : l_3 : l_4=(-a^2b-{}^2c^2):(a-bh+cg):$
$$(b-cf+ah):(c-ag+bf).$$

$$\therefore\ l_1=\frac{-(a^2+b^2+c^2)}{\mathrm{P}},\quad l_2=\frac{a-bh+cg}{\mathrm{P}}.$$

$$l_3=\frac{b-cf+ak}{\mathrm{P}},\qquad l_4=\frac{c-ag+bf}{\mathrm{P}}.$$

where $\mathrm{P}^2\equiv(a^2+b^2+c^2)(1+f^2+g^2+h^2+a^2+b^2+c^2)$

Substituting these values for l_1, l_2, l_3, l_4 in (1) we obtain, after simplification—

$$\begin{aligned}\cos\theta&=\frac{(a^2+b^2+c^2)k^2-2(a^2+b^2+c^2)(k^2-k)}{(a^2+b^2+c^2)k^2}\\&=\frac{k^2-2(k^2-k)}{k^2}\\&=\frac{2k-k^2}{k^2}\\&=\frac{2-k}{k}\\&=\frac{1-f^2-g^2-h^2-a^2-b^2-c^2}{1+f^2+g^2+h^2+a^2+b^2+c^2}\qquad\ldots\quad(2)\end{aligned}$$

where $$k=1+f^2+g^2+h^2+a^2+b^2+c^2$$

or, $$\frac{1-\cos\theta}{1+\cos\theta}=\frac{2(f^2+g^2+h^2+a^2+b^2+c^2)}{2}$$

$$=f^2+g^2+h^2+a^2+b^2+c^2$$

$$i.e., \tan^2\frac{\theta}{2}=f^2+g^2+h^2+a^2+b^2+c^2 \qquad \ldots \quad (3)$$

This formula enables us to find the angle of rotation for the axis-plane whose direction-cosines are proportional to $f, g, h, c, -b, a$.

From (3) it appears that the direction-cosines of the plane may be taken to be

$$f/\tan\frac{\theta}{2}, \quad g/\tan\frac{\theta}{2}, \quad h/\tan\frac{\theta}{2},$$

$$c/\tan\frac{\theta}{2}, \quad -b/\tan\frac{\theta}{2}, \quad a/\tan\frac{\theta}{2}.$$

We may arrive at the same result [*] from the fact that the angle of rotation depends upon a single constant. Let $f, g, h, c, -b, a$ be proportional to the direction-cosines of the axis plane, with the condition $af+bg+ch=0$.

If f', g', h' $c', -b', a'$ be the actual direction-cosines, we have

$$f:g:h:c:-b:a=f':g':h':c':-b':a';$$

$$\rho a=a', \rho b=b', \rho c=c', \rho f=f', \rho g=g', \rho h=h'.$$

$$\therefore \rho^2(f^2+g^2+h^2+a^2+b^2+c^2)=1, \text{ since } \Sigma a'^2=1.$$

Now, if we subject the quantities $f, g, h, c, -b, a$ to a further condition, the plane will be fixed. Consequently, ρ may be made to depend upon the angle of rotation, or, we may write $\rho=1/\sqrt{f^2+g^2+h^2+a^2+b^2+c^2}=\cot\frac{\theta}{2}$.

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[*] Cole has obtained the expression for θ from this latter consideration.

$\therefore$ We may take $a'=a\cot\frac{\theta}{2}$, etc.

Thus, we obtain $\tan^2\frac{\theta}{2}=f^2+g^2+h^2+a^2+b^2+c^2$

and $$\sec^2\frac{\theta}{2}=1+f^2+g^2+h^2+a^2+b^2+c^2.$$

Example : Suppose the axis plane is taken to be one of the co-ordinate planes, (say) the plane of x_1 and x_2.

$\therefore$ We have $a=b=c=g=h=0$.

Now, if the axes of x_3 and x_4 are turned through an angle θ, since the axes of x_1 and x_2 remain fixed, the formulae of transformation become—

$$x'_1=x_1,\ x'_3=\frac{1-f^2}{1+f^2}x_3+\frac{2f}{1+f^2}x_4$$

$$x'_2=x_2,\ x'_4=\frac{-2f}{1+f^2}x_3+\frac{1-f^2}{1+f^2}x_4$$

But, by considering the geometry of the plane (x_3, x_4), we have—

$$^{*}\ x'_3=\cos\theta.\ x_3-\sin\theta.\ x_4$$

$$x'_4=\sin\theta.\ x_3+\cos\theta.\ x_4.$$

$\therefore$ Comparing these two forms, we obtain

$$\cos\theta=\frac{1-f^2}{1+f^2}\ \text{and}\ \sin\theta=\frac{-2f}{1+f^2}.$$

[These results evidently follow from formula (2)].

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[*] Salmon's Conics—§ 9.

Thus we see that the angle of rotation is determined by a single constant f.

Corollary:—The expression for the angle θ of rotation for the axis-plane

$$l_1x_1+l_2x_2+l_3x_3+l_4x_4=0$$

$$l'_1x_1+l'_2x_2+l'_3x_3+l'_4x_4=0$$

is given by

$$\cos\theta=\frac{1-L_{12}^2-L_{13}^2-L_{14}^2-L_{23}^2-L_{24}^2-L_{34}^2}{1+L_{12}^2+L_{13}^2+L_{14}^2+L_{23}^2+L_{24}^2+L_{34}^2} \quad \ldots \quad (3)$$

22. Two successive simple rotations around two different axis-planes are together equivalent to a simple rotation, *i.e.*, a rotation around an axis-plane, if, and only if, the two axis-planes intersect in a line.

Let us define the two rotations by the following formulae of transformation :—

$$\left.\begin{aligned} x_i' &= a_{i1}x_1+a_{i2}x_2+a_{i3}x_3+a_{i4}x_4 \\ x_i'' &= a'_{i1}x'_1+a'_{i2}x'_2+a'_{i3}x'_3+a'_{i4}x'_4. \end{aligned}\right\} \quad \ldots \quad (1)$$

$(i=1, 2, 3, 4.)$

and suppose that the rotations occur in the order in which they are written. We therefore have

$$\begin{aligned} x_i'' &= a'_{i1}(a_{11}x_1+a_{12}x_2+a_{13}x_3+a_{14}x_4) \\ &\quad +a'_{i2}(a_{21}x_1+a_{22}x_2+a_{23}x_3+a_{24}x_4) \\ &\quad +a'_{i3}(a_{31}x_1+a_{32}x_2+a_{33}x_3+a_{34}x_4) \\ &\quad +a'_{i4}(a_{41}x_1+a_{42}x_2+a_{43}x_3+a_{44}x_4) \\ &= a''_{i1}x_1+a''_{i2}x_2+a''_{i3}x_3+a''_{i4}x_4 \quad \ldots \quad (2) \end{aligned}$$

where $a''_{ij}=a'_{i1}a_{1j}+a'_{i2}a_{2j}+a'_{i3}a_{3j}+a'_{i4}a_{4j}$

$(i=j=1, 2, 3, 4.)$

Let a, b, c, f, g, h ; and a', b', c', f', g', h' be the independent constants in the above two transformations and $a'', b'', c'', f'', g'', h''$ be those of their resultant.

Also, let Δ, Δ' and Δ'' be the corresponding functions of the two component transformations and their resultant respectively.

Then we have *

$$\left.\begin{aligned}
Da'' &= a+a'-\Delta' f-\Delta f'+bh'-b'h+c'g-cg', \\
Db'' &= b+b'-\Delta' g-\Delta g'+cf'-c'f+a'h-ah', \\
Dc'' &= c+c'-\Delta' h-\Delta h'+ag'-a'g+b'f-bf', \\
Df'' &= f+f'-\Delta' a-\Delta a'+bc'-b'c+gh'-g'h, \\
Dg'' &= g+g'-\Delta' b-\Delta b'+ca'-c'a+f'h-fh', \\
Dh'' &= h+h'-\Delta' c-\Delta c'+ab'-a'b+fg'-f'g. \\
\therefore D\Delta'' &= \Delta+\Delta'+a'f+af'+bg'+b'g+ch'+c'h,
\end{aligned}\right\} \quad \dots \quad (3)$$

where $\qquad D=1+\Delta\Delta'-aa'-bb'-cc'-ff'-gg'-hh'$.

It is easily seen that $\Delta''=a''f''+b''g''+c''h''$.

Now, since each of the given rotations leaves a plane fixed, we must have $\Delta=0$ and $\Delta'=0$ (§ 19)

If the resultant rotation is to be a simple one, *i.e.*, if it leaves a plane fixed, we must have $\Delta''=0$, and consequently, from the last of formulae (3) it follows that we must have $af'+a'f+bg'+b'g+ch'+c'h=0$. But the last condition implies that the two axis-planes should intersect in a line (§ 15). Thus, if the resultant of two simple rotations is also a simple rotation, the necessary condition is that the two axis-planes of the two component rotations must intersect in a line.

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* These have been calculated by Cole—American Journal of Mathematics, Vol. XII, 1890; or, we may directly calculate them by applying Prof. Cayley's formulae.

Again, if in addition to the conditions $\Delta=0$, $\Delta'=0$, we have further $af''+a'f+bg'+b'g+ch'+c'h=0$, if follows that $\Delta''=0$, or the resultant rotation leaves a plane fixed, *i.e.*, if the axis planes of two component rotations intersect in a line, the resultant is a simple rotation.

Thus the resultant * of two simple rotations is not in general a simple rotation ; it is a simple rotation when, and only when, the axis planes of the two component rotations have a common line of section. This condition is both necessary and sufficient.

We may therefore enunciate the converse theorem as follows :—If two successive simple rotations are together equivalent to a simple rotation, the axis planes of the two rotations are in a three-way space ; and when they have a point in common, they intersect in a line.

23. The axis-plane of the resultant rotation passes through the line of intersection of the two given axis planes ; or the three axis-planes are all perpendicular to one and the same plane.

As before, let the direction-cosines of the axis-plane be

$$a,\ b,\ c,\ f\ldots\ ;\quad a',\ b',\ c',\ f',\ldots;\quad a'',\ b'',\ c'',\ f'',\ldots$$

Since the resultant is a simple rotation, we have—

$$af'+a'f+bg'+b'g+ch'+c'h=0.$$

$\therefore$ The first and the second axis-planes intersect.

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* In contradistinction to the name "simple rotation," the name "double rotation" has been given to the resultant of two rotations.

Also,
$$\begin{aligned}
&D(af''+a''f+bg''+b''g+ch''+c''h)\\
&=(af'+a'f+bh'f+b'hf+c'fg-cfg')\\
&+(af+af'+abc'-ab'c+agh'-ag'h)\\
&+(bg+b'g+cf'g-c'fg+a'gh-agh')\\
&+(bg+bg'+a'bc-abc'+bf'h-bfh')\\
&+(ch+c'h+ag'h-a'gh+b'fh-bf'h)\\
&+(ch+ch'+ab'c-a'bc+cfg'-cf'y)\\
&=2(af+bg+ch)+(af'+a'f+bg'+b'y+ch'+c'h)\\
&=0.
\end{aligned}$$

$\therefore$ The third axis-plane intersects the first one in a line.

Similarly, $a'f''+b'g''+c'h''+a''f'+b''g'+c''h'=0.$

Thus the three axis-planes intersect two and two in a line.

If these planes have one other point common besides the origin, they intersect in a line.

Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be the line of intersection of the planes.

Then,
$$\Sigma\lambda_i l_i=0 \qquad \Sigma\lambda_i l'_i=0, \qquad \Sigma\lambda_i m_i=0$$
$$\Sigma\lambda_i m'_i=0, \quad \Sigma\lambda_i n_i=0, \quad \Sigma\lambda_i n'_i=0.$$

By eliminating $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in turn from these equations, we obtain four equations of condition :

Thus, eliminating λ_4 from each pair we obtain

$$\left.\begin{aligned}
h\lambda_1-b\lambda_2+a\lambda_3&=0\\
h'\lambda_1-b'\lambda_2+a'\lambda_3&=0\\
h''\lambda_1-b''\lambda_2+a''\lambda_3&=0
\end{aligned}\right\}$$

$$\therefore \begin{vmatrix} h & b & a \\ h' & b' & a' \\ h'' & b'' & a'' \end{vmatrix}=0 \qquad \dots \quad (1)$$

Similarly we obtain

$$\begin{vmatrix} b & c & f \\ b' & c' & f' \\ b'' & c'' & f'' \end{vmatrix} = 0 \quad \ldots \quad (2)$$

$$\begin{vmatrix} c & a & g \\ c' & a' & g' \\ c'' & a'' & g'' \end{vmatrix} = 0 \ldots (3) \, ; \quad \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} = 0 \quad \ldots \quad (4)$$

If theses four conditions are satisfied by the direction-cosines of the three axis-planes, they all have a common line of section.

But if these conditions are not satisfied, it is possible to determine six quantities $a''', b''', c''', f''', g''', h'''$ such that they satisfy the following seven relations :—

$$\left.\begin{array}{l} af''' + bg''' + ch''' + fa''' + gb''' + hc''' = 0 \\ aa''' + bb''' + cc''' + ff''' + gg''' + hh''' = 0 \\ a'f''' + b'g''' + c'h''' + f'a''' + g'b''' + h'c''' = 0 \\ a'a''' + b'b''' + c'c''' + f'f''' + g'g''' + h'h''' = 0 \\ a''f''' + b''g''' + c''h''' + f''a''' + g''b''' + h''c''' = 0 \\ a''a''' + b''b''' + c''c''' + f''f''' + g''g''' + h''h''' = 0 \end{array}\right\}$$

$$a'''f''' + b'''g''' + c'''h''' = 0.$$

Hence, the three axis-planes either intersect in a line or are all perpendicular to one and the same plane.

24. To find the resultant of two simple rotations whose axis-planes are absolutely perpendicular to each other.

Let the direction-cosines of one axis-plane be L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}, and consequently those of its absolutely perpendicular axis-plane be L_{34}, L_{42}, L_{23}, L_{31}, L_{12}.

Then the constants of transformations are given by

$a=k\ L_{12},\ b=k\ L_{13},\ c=k\ L_{14},\ f=k\ L_{34},-g=k\ L_{24},$

$h=k\ L_{23}.$

and $a'=k'\ L_{34},\ b'=k'\ L_{42},\ c'=k'\ L_{23},\ f'=k'\ L_{12},$

$g'=-k'\ L_{31},\ h'=k'\ L_{14},$

where k and k' are any two multipliers.

Then, $\Delta''=af'+a'f+bg'+b'g+ch'+c'h$

$=kk'\{\overset{2}{L}_{12}+\overset{2}{L}_{13}+\overset{2}{L}_{14}+\overset{2}{L}_{42}+\overset{2}{L}_{23}+\overset{2}{L}_{34}\}$

$=kk'.$

$D=1-aa'-bb'-cc'-ff'-gg'-hh'$

$=1-2kk'\{L_{34}.L_{12}+L_{13}.L_{42}+L_{14}.L_{23}\}$

$=1$, since $L_{34}.L_{12}+L_{13}.L_{42}+L_{14}.L_{23}=0.$

Also, $\Delta=\Delta'=0;$

$\therefore\ a''=a+a'+bh'-b'h+c'g-cg'$

$=a+a'+kk'\{L_{13}.L_{14}-L_{42}.L_{23}+L_{23}.L_{42}-L_{14}.L_{13}\}$

$=a+a'=k\ L_{12}+k'\ L_{34}.$

Similarly, $b''=b+b'=k\ L_{13}+k'\ L_{42}$

etc. etc.

From the above values of a'', b'', c'', etc., it appears that they are symmetrical as regards the elements of the two axis-planes. Hence we may enunciate the following theorem :—

Rotations around two absolutely perpendicular planes are commutative; after two such rotations all points of the Hyper-space take the same positions, whichever rotation comes first.

25. We have seen that in a simple rotation the Hyper-sphere is transformed into itself. Therefore, when it is rotated successively around two absolutely perpendicular axis-planes through its centre, it will be transformed into itself. Thus, we obtain the theorem :—Any position of a Hyper-sphere can be obtained from any other position with the same centre by a simple or a double rotation.*

Parallel motion:—When the two rotations of a double rotation are equal, it is a parallel motion, corresponding to an isocline rotation † at the centre of the Hyper-sphere. In a parallel motion all great circles parallel ‡ to the circles of rotation in the sense of the rotation rotate on themselves, and the motion can be regarded as a parallel motion along any polar pair taken from this set of circles. The motion can be regarded as a parallel motion with regard to the set of circles without any reference to the axis-planes or planes of rotation.

26. We have seen that a Hyper-sphere can be moved freely on itself, which readily follows from the fact that a Hyper-sphere is always converted into itself by a simple or a double rotation. Consequently, if we take a portion of its surface (which is of three dimensions) and move it freely along the surface, it will always wholly coincide with the surface itself. Therefore we can say that a Hyper-sphere is a space of constant curvature.§ If the space in which we live were a Hyper-sphere in Euclidean space of four dimensions, we should realise what is called the Elliptic Geometry. The Elliptic Geometry assumes that our space is a space of constant curvature like a Hyper-sphere, and not a space of no curvature like a three-way space. The Geometry of the Hyper-sphere is found to be the same as the Double Elliptic Geometry.* We shall take up the subject of curvature of surfaces on another occasion.

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* Manning—loc. cit. Th. I, p. 219.

† To be explained later.

‡ Clifford—Proc. London Mathematical Society, Vol IV (1873).

§ Gauss—Coll. Works.

27. The motion in Hyper-space is completely determined by that of its four non-coplanar points; *i.e.*, if we are given any two positions of a figure such that one can be obtained from the other by a motion in Hyper-space, then, any motion in Hyper-space which brings four non-coplanar points from their first to their second positions will carry every point of the figure from its first to the second position.

For, each point of the three-way space determined by the four given points comes to its second position by this motion; and any point which does not belong to this three-way space remains at the same distance from it, on the same side of it, and with the same projection upon it. Therefore it must come to its second position.

Or, we may prove it by the analytical method as follows:—

Let the motion be determined by the following transformation scheme:—

$$x'_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4 + a_{i4}$$

$$(i = 1, 2, 3, 4.)$$

The scheme involves twenty independent constants. The condition that any four points transform to four others is equivalent to sixteen relations among these constants. Further, the fact that the four points are non-coplanar supplies four additional conditions, namely,—the conditions for the constancy of the mutual distances of the four points (non-coplanar). Therefore, the twenty constants are connected by twenty equations of conditions and can therefore be uniquely determined. The scheme is therefore unique and will transform any point to the corresponding point in its new position.

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* Manning—Non-Euclidean Geometry.

28. If after a motion in Hyper-space, three non-collinear points of a figure remain fixed, then every point of the figure with remain fixed, or will be rotated through a certain angle around the plane of the three points.

Suppose the origin is one of these three fixed points. Then the scheme becomes—

$$\left.\begin{aligned} x'_1 &= a_1x_1+a_2x_2+a_3x_3+a_4x_4 \\ x'_2 &= b_1x_1+b_2x_2+b_3x_3+b_4x_4 \\ x'_3 &= c_1x_1+c_2x_2+c_3x_3+c_4x_4 \\ x'_4 &= d_1x_1+d_2x_2+d_3x_3+d_4x_4 \end{aligned}\right\} \quad \cdots \quad (1)$$

If besides the origin two other points remain fixed, we must have the following equations satisfied by more than one set of independent values of the variables :—

$$\left.\begin{aligned} (a_1-1)x_1+a_2x_2+a_3x_2+a_4x_4&=0 \\ b_1x_1+(b_2-1)x_2+b_3x_3+b_4x_4&=0 \\ c_1x_1+c_2x_2+(c_3-1)x_3+c_4x_4&=0 \\ d_1x_1+d_2x_2+d_3x_3+(d_4-1)x_4&=0. \end{aligned}\right\} \quad \cdots \quad (2)$$

Then, each of these equations may be an identity, and in that case, we must have $a_1=1$, $b_2=1$, $c_3=1$, $d_4=1$ and all other coefficients zero. The scheme becomes $x_1=x'_1$, $x_2=x'_2$, $x_3=x'_3$ and $x_4=x'_4$, which is a scheme in which all the points remain fixed.

If the equations in (2) are not identically satisfied, we obtain after eliminating the variables the condition that $\triangle$ should be identically zero. This case we have investigated in § 19, and we have seen there that the transformation amounts to a rotation about the plane determined by the equations (2), which evidently passes through the three given points.

29. If after a motion in Hyper-space, two points of a figure remain fixed, then every point of the figure remains fixed or is rotated through a certain angle about a certain plane passing through the points.

Supposing that the origin is one of the fixed points the scheme of transformation is given by (1) of the previous article. If one other point is to remain fixed, equations (2) must be satisfied by values other than 0, 0, 0, 0. Then we must have either the coefficients in (2) separately zero, in which case all the points remain fixed after transformation, or $\triangle = 0$, and this determines a rotation about a plane through the points.

Corollary:—If after a motion in Hyper-space, one point of a figure remains fixed, then every point of it will remain fixed or will undergo a single or double rotation.

This follows from the scheme (1) of § 18.

30. Rotation in a Hyper-space of r-dimensions :— From what has been said above it is easily seen that we can generalise the theorems and can extend the formulae to spaces of any number of dimensions. The scheme of transformation for any r-way space contains $r(r+1)$ constants. The absolute terms will vanish if the origin is to remain fixed after transformation. The scheme then contains r^2 independent constants which, according to Cayley's formulae, can be expressed in terms of $\frac{1}{2}r(r-1)$ others. If other points besides the origin are to remain fixed, a determinant of the rth order of the coefficients must vanish, whose value can be determined in terms of the $\frac{1}{2}r(r-1)$ constants of transformation. In general, other points of the r-way space do not remain fixed, but if this determinant $\triangle$ vanishes, other points remain fixed and all these points determine an $(r-2)$-way space, which we may call the axis $(r-2)$-way space, *i.e.*, in this motion the $(r-2)$-way space remains fixed and all other points are supposed to rotate about this $(r-2)$-way space as fixed. The constants in transformation may be shown to be proportional to the direction-cosines of the $(r-2)$-way space, if we extend the notion of direction-cosines of a k-way space and define them to be the "cosine-products" of the angles between the k-way space and the axial k-way spaces. The number of such products is $rC_{r-2}\, rC_2=\frac{1}{2}r(r-1)$ and these are proportional to the constants in transformation.

In the same way, we may widen our conceptions and obtain formulae corresponding to rotations in Hyper-spaces of any number of dimensions.

31. Planes isocline to a given plane:—Let α and β be any two planes intersecting at a point O. Let $l, m; p, q$ be the minimal lines respectively in these two planes, so that the planes (l, p) and (m, q) are the two common perpendicular planes to α and β, and they are absolutely perpendicular to each other.

If θ and θ' be the angles between the two planes α and β, we have

$$\widehat{lp}=\theta \text{ and } \widehat{mq}=\theta'.$$

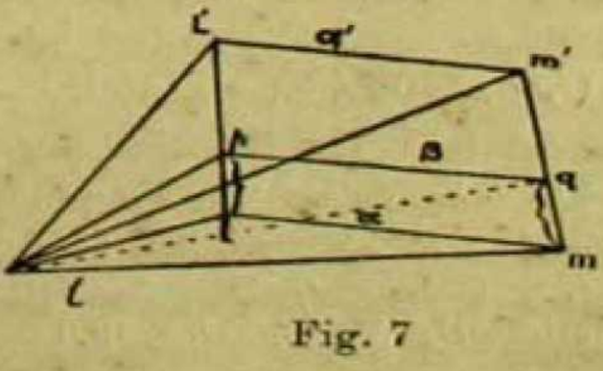

Fig. 7

Since the planes (l, p) and (m, q) intersect the plane α, they both intersect the absolutely perpendicular plane α'. Let l' and m' be these lines of intersection with the absolutely perpendicular plane. Thus we have a system of four mutually perpendicular lines (l, m, l', m'). Any three of these determine a three-way space perpendicular to the fourth. Without disturbing the fourth we may permute these three in a cyclic order by a rotation about the line in their three-way space equally inclined to them. In this way we obtain twelve different arrangements of the four lines. The system is said to be congruent * to itself in any of these twelve arrangements. But if we rotate the system of three in the four-way space about a certain axis-plane, the fourth line will be disturbed and the system will not completely occupy its former position. For instance, if we take the plane through l bisecting the angle between l' and m as the axis-plane, the direction of m' will be reversed, if we rotate the system through 180° around this plane. In fact m' will occupy a position symmetrical with respect to the three-way space of l, l', m.

Then, β makes with α angles θ and θ' and thereby we associate a sense of rotation in β corresponding to the order l, m, *i.e.*, a sense of rotation which turns p through 90° to the position of q. When $\theta = \theta'$, β is isocline to α as well as to α'. By giving different values to θ, we obtain an infinite number of planes isocline to α and α' and to one another. All these planes are perpendicular to (l, l') and (m, m') and are called a *series of isocline*

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* C. J. Keyser,—"Concerning angles and angular determination of planes in four-space."—Bull. of the American Math. Soc., Vol. 8, 1902.

I. Stringham—On the Geometry of planes, etc., Trans. of the American Math. Soc., Vol. 2, 1901.

planes. The planes (l,l') and (m,m') are mutually absolutely perpendicular and also isocline to each other. Thus we obtain a series of planes isocline to (l, l') and (m, m'). This latter series is called "*a series of isocline planes conjugate to the former series.*" It is to be noticed that the planes of the one series are all perpendicular to the planes of the conjugate series.

32. Isocline rotation :—There are other systems of planes isocline to the plane α. Apply a rotation to the system in which the plane α remains fixed ; *i.e.*, let us rotate the system around α as an axis-plane. Then the plane α' turns on itself through a certain angle (angle of rotation), *i.e.*, the lines l' and m' are rotated through a certain angle. Denote by l'' and m'' these lines in their new position. We obtain in this way a new system of isocline planes corresponding to this new arrangement, together with a corresponding conjugate series. The latter series of conjugate planes are all perpendicular to *all* planes of the new series, but not to *any* of the former series except α and α'. We may here establish the following theorem :—

Theorem :—If a series of planes, isocline in the same sense to a given plane, is rotated around the given plane as axis-plane, they still remain isocline to the axis-plane and the conjugate series are turned through the angle of rotation about the lines in which they intersect the given plane.

Let us take the series of planes isocline in the same sense to the axial plane of x_1 and x_2. Let (l, m) be the minimal lines of a plane of the series and (x_1, x_2) those of the axis-plane, and let the minimal planes* intersect the plane (x_3, x_4) in the lines l' and m'. Then the plane (l, l') belongs to the conjugate series. In the rotation, the axis of x_1 remains fixed, while l' is turned through a certain angle θ in the plane (x_3, x_4) to a new position l''. Let p and q be the new positions of l and m. It is required to prove that (p, q) is isocline to the axis-plane and the plane of (x_1, p, l'') is perpendicular to (x_1, x_2) and the angle between this plane and that of (x_1, l, l'') is θ, *i.e.*, the angle of rotation.

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* *Cf.* Ana. Geo., Part I, § 33.

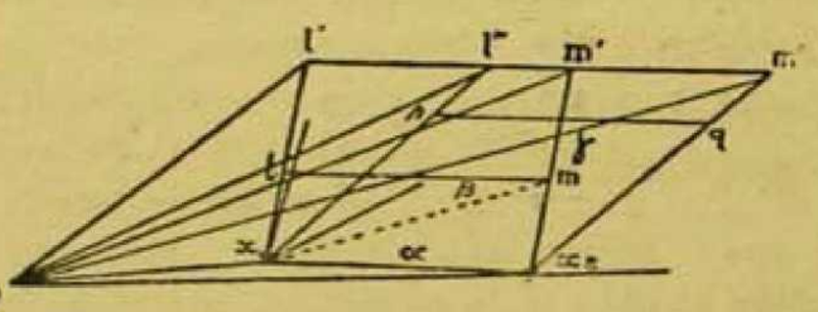

Fig. 8

The formulae of rotation for this are, by § 21,—

$$\left.\begin{aligned} x_1 = x'_1,\ x_2 = x'_2,\ x'_3 &= \frac{1-f^2}{1+f^2}\, x_3 + \frac{2f}{1+f^2}\, x_4 \\ x'_4 &= \frac{-2f}{1+f^2}\, x_3 + \frac{1-f^2}{1+f^2} \cdot x_4. \end{aligned}\right\}$$

Evidently, the lines l' and m' are the axes of x_3 and x_4 respectively. The direction-cosines of l and m may be taken to be respectively $(l_1, 0, l_3, 0)$ and $(0, m_2, 0, m_4)$.

If l is a point on the line l, it is transformed into a point p on the line p, where $kp_1 = x'_1 = x_1$; $kp_2 = x'_2 = x_2 = 0$.

$kp_3 = x'_3 = \dfrac{1-f^2}{1+f^2}\, l_3$; and $kp_4 = x'_4 = \dfrac{1-f^2}{1+f^2} . l_3$, where k is arbitrary.

$\therefore$ The direction-cosines of p are proportional to—

$$\left(l_1,\ 0,\ \frac{1-f^2}{1+f^2}\, l_3,\quad \frac{-2f}{1+f^2}\, l_3\right).$$

Similarly, those of the line q are

$$\left(0,\ m_2,\ \frac{2f}{1+f^2}\, m_4,\ \frac{1-f^2}{1+f^2} m_4\right)$$

$\therefore$ The lines p and q are mutually perpendicular, as can easily be verified.

Also, the planes (x_1, p) and (x_2, q) are perpendicular to the plane (x_1, x_2).

Thus (x_1, p) and (x_2, q) are the minimal planes of (x_1, x_2) and (p, q).

$$\cos \widehat{px_1} = \frac{l_1}{\sqrt{l_1^2 + \left(\frac{1-f^2}{1+f^2}\right) l_3^2 + \frac{4f^2}{(1+f^2)^2}\, l_3^2}}$$

$$= \frac{l_1}{\sqrt{l_1^2 + l_3^2}} = l_1, \text{ since } l_1^2 + l_3^2 = 1.$$

Similarly, $\cos \widehat{qx_2} = m_2$.

But (l, m) is isocline to (x_1, x_2). $\therefore \quad l_1 = m_2$.

Hence also (p, q) is isocline to (x_1, x_2) and the sense is also the same. Thus the plane (l, m) remains isocline in the same sense to the axis-plane even after rotation.

Again, the angle between the planes of (l, x_1) and (p, x_1) is given by—

$$\cos \phi = \frac{(lp) - (lx_1)(px_1)}{[lx_1][px_1]}$$

$$\text{or, } \cos\phi = \frac{(lp)-(lx_1)^2}{[lx_1]^2} = \frac{(lp)-\cos^2\lambda}{\sin^2\lambda}, \text{ where } \lambda = \widehat{lx_1}.$$

Now,

$$(lp = l_1^2 + \frac{1-f^2}{1+f^2} \cdot l_3^2$$

$$\therefore \cos\phi = \frac{l_1^2 + \frac{1-f^2}{1+f^2} \cdot l_3^2 - l_1^2}{l_2^2} = \frac{1-f^2}{1+f^2}$$

$$= \cos\theta, \text{ where } \theta \text{ is the angle of rotation.}$$

Therefore, the angle ϕ being constant, the conjugate series are turned through the same angle θ.

It is to be noticed here that the conjugate series $\alpha-\beta$ and the conjugate series to $\alpha-\gamma$ have one and only one pair of common perpendicular planes, namely α and α', and these two series are isocline in opposite senses. This can be proved as follows :—If the two series are to have a common plane, that plane must be isocline to (l, x_1) as well as to (p, x_1). Let such a plane intersect (x_1, x_2) in a line 'a' and $(l,' m')$ in a line 'b,' b lying between l and $l.''$ Then $\widehat{x_1a} = \widehat{l'b}$, and $\widehat{x_1a} = \widehat{l''b}$. Thus $\widehat{l'b} = \widehat{lb''}$, so that the line b bisects the angle between l' and l''. The plane (a, b) is then isocline to (l, x_1) in a sense opposite to that in which it is isocline to (p, x_1).

Hence the two series are isocline to the plane (a, b) through the bisector of the angle $\widehat{l'l''}$ and also to its absolutely perpendicular plane $(a'b')$. It further appears that the planes (x_1, x_2) and (l', m') are the only planes which are perpendicular to both the series and also to the common pair of planes (a, b) and (a', b').

Thus we obtain the theorem :—If two planes are isocline to a third in opposite senses, the three planes have one and only one pair of common perpendicular planes.

From what has been said above, we may at once deduce the following as a corollary :—If two planes are isocline to a third in opposite senses and make the same angle with it, they always intersect. For, the planes (l, x_1) and (p, x_1) which are isocline to (a, b) in opposite senses, and make the same angle $\widehat{x_1 a}$ with it, intersect in the line x_1 on the plane (x_1, x_2).

33. From the preceding article it is easily seen that the planes (l, m) and (p, q) are isocline to each other in the same sense in which each of them is isocline to the plane (x_1, x_2).

$$\text{For, } (lp)=l_1^2+\frac{1-f^2}{1+f^2}.l_3^2 \text{ and } (mq)=m_2^2+\frac{1-f^2}{1+f^2}m_4^2$$

Also, $l_1=m_2$ and $l_3=m_4$.

$\therefore$ $(lp)=(mq)$ and consequently the two planes are isocline and in the same sense.

We may prove this theorem geometrically as follows :— Let the two planes β and γ be isocline to the plane α in the same sense. Then, β and γ have one pair of common perpendicular planes and let them intersect β and γ respectively in the lines l, l' and m, m'.

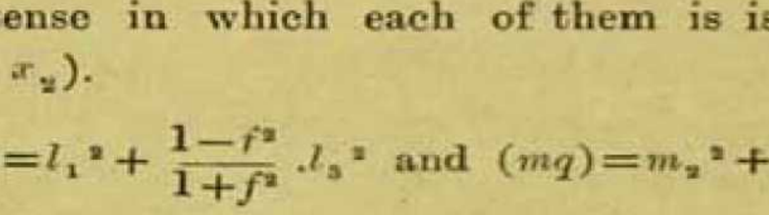

Fig. 9

Now, $\widehat{ll'}$ and $\widehat{mm'}$ are two right angles in the two planes β and γ. In the plane α take two lines p and p', such that (l, p) and (l', p') are two common perpendicular planes to α and β; and let q and q' be two perpendicular lines in γ, such that (p, q) and $(p,' q')$ are two common perpendicular planes to α and γ.

Thus $\widehat{lp}$ = the isoclinic angle between α and β, and $\widehat{pq}$ = the isoclinic angle between α and γ.

Now, since $\widehat{ll'}$ and $\widehat{mm'}$ are right angles, $\widehat{pp'}$ and $\widehat{qq'}$ are also right angles. Therefore $\widehat{mq}=\widehat{m'q'}$.

Since β and γ are isocline to α in the same sense, the two planes (l, p) and (q, p) are inclined at a constant angle, so that the angle between (l, p) and (p, q) is equal to the angle between (l', p') and (p', q'). Now, consider the solid angles lpq and $l'p'q'$. We have $\widehat{lp}=\widehat{l'p'}$, $\widehat{pq}=\widehat{p'q'}$ and also the dihedral angles along p and q are equal.

$\therefore$ The third face angles $\widehat{lq}$ and $\widehat{l'q'}$ are equal.*

Again, consider the solid angles lmq and $l'm'q'$. We have $\widehat{mq}=\widehat{m'q'}$ and $\widehat{lq}=\widehat{l'q'}$; also the dihedral angles along m and m' are right angles.

Therefore, the angle $\widehat{lm}$ = angle $\widehat{l'm'}$, *i.e.*, the planes β and γ are isocline, and in the same sense as to α. For, if not, α and β, being isocline to γ in opposite senses, would have with γ one pair of common perpendicular planes perpendicular to all three. But

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* This can be deduced as a corollary from §. 27, Ana. Geo., Part I.

β and γ being isocline to α in the same sense have no common perpendicular planes with α, unless they belong to the same series of isocline planes.

N.B.—When we say that two planes are isocline to a third, it is understood that they do not generally belong to the same series of isocline planes.

The above theorem may be stated as follows :—If two planes are isocline to a third in the same sense, they are isocline to each other in this sense also.*

34. Any plane polygon is similar to its projection on an isocline plane.

Let the two planes intersect at O ; and A and B be two points of the polygon. Let A′ and B′ be the (orthogonal) projections of A and B respectively. Then, we have two triangles OAA′ and OBB′. The planes being isocline, $\angle AOA' = \angle BOB'$ and $\angle AA'O = \angle BB'O =$ a right angle. $\therefore$ The triangles are similar, and consequently $OA{:}OA' = OB{:}OB'$ and $\angle AOB = \angle A'OB'$. $\therefore$ The triangles AOB and A′OB′ are similar. Hence, any triangle is similar to its projection on an isocline plane.

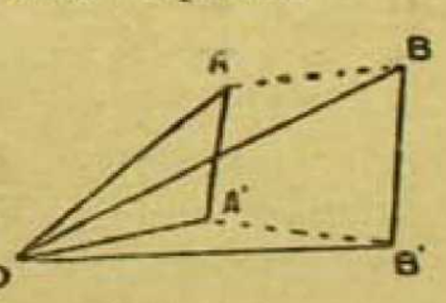

Fig 10

Now if we take a point in the plane of the polygon and join it to the vertices, the polygon is divided into a number of triangles which are similar to their projections on the isocline plane. Hence, the two polygons are similar. Further, if $\triangle$ be the area of the plygon and θ be the isoclinic angle, then the area of the polygon of projection is $\triangle\cos^2\theta$; and in general if θ and θ' be the minimum angles between two planes, the area of the projection of polygon $=\triangle\cos\theta.\ \cos\theta'$.

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* *Cf.* Manning—loc. cit. §. 109.

Corollary :—Taking a polygon inscribed in a circle, if we increase the number of sides indefinitely, the circle may be regarded as the limiting form of the polygon and we obtain the theorem :—*The projection of a circle upon an isocline plane is also a circle.*

35. The converse theorem is also true *i.e.*, if a plane polygon is similar to its projection upon another plane, the two planes are isocline, and when the length of a side is equal to its projection, the two planes are parallel.

When the two planes are parallel, the two polygons may be regarded as the projections of one and the same polygon, for the planes projecting a figure upon one of two parallel planes project the same figure upon the other and the two projections are equal. We may suppose then that the plane of projection passes through a vertex of the given polygon. Now, consider the two common perpendicular planes of the two given planes. They intersect the planes in two pairs of lines, the angles between which measure the minimum angles. From the figure of the preceding article, we have the two triangles AOB and A′OB′ similar.

$\therefore$ AO : OA′ = OB : OB′ and $\angle$ˢ OA′A and OB′B are right angles. $\therefore$ $\triangle$ˢ AOA′ and BOB′ are similar and consequently $\angle$ AOA′ = $\angle$ BOB′. Thus the two planes are isocline, which also follows from the fact that if the projecting planes through the origin (one of the vertices of the original polygon) intersect the planes forming similar triangles. Hence the two planes are isocline.*

36. Two conjugate series of isocline planes determine a pair of absolutely perpendicular planes to which the planes of the two series are isocline in opposite senses, all inclined at an angle of 45° with them.

Let us take two conjugate series of planes of the type $\alpha\ (l, m)$ and $\beta\ (l, l')$. Then there are two (mutually absolutely perpendicular) planes which are orthogonal to both α and β. One of these planes (q, q') passes through the line l, *i.e.* the common line of α and β, and the other is $l'm$.

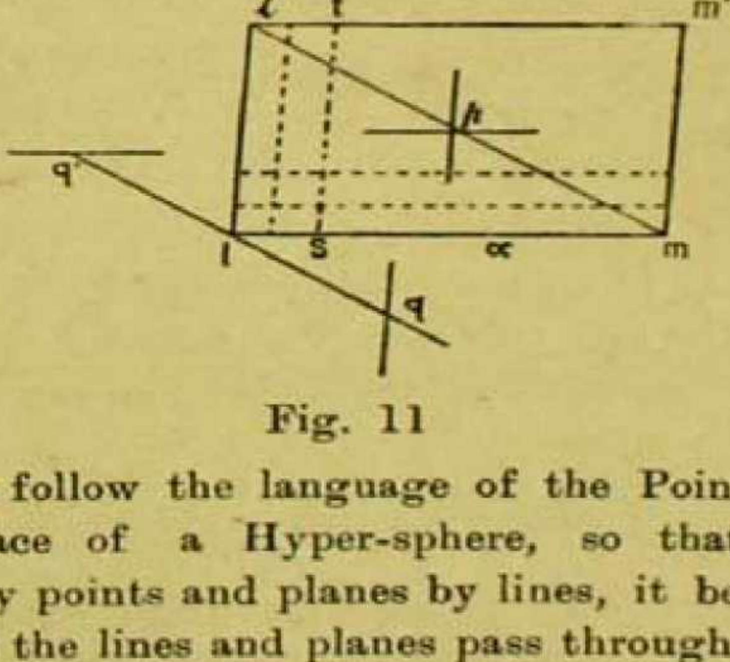

Fig. 11

[Here we follow the language of the Point-Geometry on the surface of a Hyper-sphere, so that lines are represented by points and planes by lines, it being understood that all the lines and planes pass through the origin and we omit the mention of O.]

Bisect the angle $\widehat{l'm}$ at p. On qq' measure angle $\widehat{lq}=\widehat{l'p}=45°$. Then p and q determine the required plane, and if we make $\widehat{lq'}=45°$, then the plane (p, q') is the other plane perpendicular to (p, q).

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* Manning—loc. cit. Cor. §. 69.

Choose l, m, l', m', for the co-ordinate axes. The direction-cosines of p are

$$p_i = \frac{l_i + m_i}{2}\ , \ i.e. \left(0,\ \frac{1}{\sqrt{2}}\ ,\ \frac{1}{\sqrt{2}}\ ,0\right)$$

Those of q are q_i with the condition $q_1 = \frac{1}{\sqrt{2}}$ and $\Sigma l'_i q_i = 0$ *i.e.*, $q_3 = 0$, and $\Sigma\ m_i q_i = 0$ *i.e.*, $q_2 = 0$. Also $\Sigma\, q_i^2 = 1$ *i.e.*, $\frac{1}{2} + q_4^2 = 1$ $\therefore\ q_4 = \frac{1}{\sqrt{2}}$;

i.e., the direction-cosines of q are $\left(\frac{1}{\sqrt{2}}\ ,\ 0,\ 0,\ \frac{1}{\sqrt{2}}\right)$.

The angle θ between the planes (l, m) and (p, q) is given by

$$e_1^2 = \cos^2\theta_1 . \cos^2\theta_2 = [lm/pq]^2 = \tfrac{1}{4}.$$

$$\therefore\ e_1 = \cos\theta_1 .\ \cos\theta_2 = \tfrac{1}{2}.$$

Similarly, $e_2 = \sin\theta_1 .\ \sin\theta_2 = \frac{1}{2}$.

$\therefore e_1 + e_2 = 1$, and the planes are isocline[*] *i.e.*, $\theta_1 = \theta_2 = \theta = 45°$.

This is also geometrically evident, for (l', m) and (q, q') are both perpendicular to (l, m), and $\widehat{pm} = \widehat{lq} = 45°$.

Similarly, (p, q) is isocline to (l, l') at an angle of 45°.

It remains to be proved that any plane of the two systems is isocline to (p, q) at an angle of 45°.

Take any line s in (l, m) such that $(ls) = s_1$ and $(ms) = s_2$, so that $s_1^2 + s_2^2 = 1$. Take any other line t in the plane (l', m'), so that $\widehat{l't} = \widehat{ls}$, such that $t_3 = -s_1$ and $t_4 = s_2$.

Thus the lines are $(s_1, s_2, 0, 0)$ and $(0, 0, -s_1, s_2)$.

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[*] *Cf.* Cor. 3, §. 24. Part I.

Now the angle between (*s*, *t*) and (*p*, *q*) is determined as follows :—

$$e_2 = \sin\theta_1 . \sin\theta_2 = \begin{vmatrix} s_1 & s_2 & 0 & 0 \\ 0 & 0 & -s_1 & s_2 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= s_1 . \frac{1}{\sqrt{2}} \cdot \left(s_1 . \frac{1}{\sqrt{2}} \right) - s_2 . \frac{1}{\sqrt{2}} \cdot \left(-s_2 . \frac{1}{\sqrt{2}} \right)$$

$$= \tfrac{1}{2}\left(s_1{}^2 + s_2{}^2\right) = \tfrac{1}{2}.$$

Also, $e_1 = \cos\theta_1 . \cos\theta_2 = [st/pq]$

$$\left| \begin{array}{cccc|cccc} s_1 & s_2 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -s_1 & s_2 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{array} \right|$$

$$= \begin{vmatrix} s_1 & 0 \\ 0 & s_2 \end{vmatrix} \begin{vmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{vmatrix} + \begin{vmatrix} s_1 & 0 \\ 0 & s_2 \end{vmatrix} \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= \tfrac{1}{2}\left(s_1{}^2 + s_2{}^2\right) = \tfrac{1}{2}.$$

$$\therefore\ e_1 + e_2 = \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

Therefore the plane (*s*, *t*) is isocline * to (*p*, *q*) inclined at an angle 45°.

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* §. 24, Cor. Part I.

Similarly, any plane of the conjugate series is inclined to (p, q) at a constant angle of 45°.

37. When two rotations around two absolutely perpendicular planes α and α' are equal, all the planes isocline in the sense corresponding to the rotation rotate on themselves, the series conjugate to any series of these planes moving as a series on itself. Every line rotates in a plane isocline to α, and any one of these planes and its absolutely perpendicular plane can be regarded as the axis-planes. This is called an "isocline rotation" and the common angle of rotation is called the "angle of rotation."

We may then enunciate the following theorem :—

In an isocline rotation every plane, not rotating on itself, remains isocline to itself in the sense opposite to the rotation.

Let β be any plane which does not rotate on itself and let β' be its new position. Let l and m, two lines in β, be rotated to the positions l' and m' in β'. But each of l and m rotate in an isocline plane, and therefore the planes (l, l') and (m, m') are isocline, and also $\widehat{ll'} = \widehat{mm'}$. Thus, we have two isocline planes (l, l') and (m, m') in which the lines have been taken, such that $\widehat{ll'} = \widehat{mm'}$. Therefore, the planes (l, m) and (l', m') are isocline in a sense opposite to that in which (l, l') and (m, m') are isocline,* *i.e.*, the planes β, β' are isocline in the sense opposite to the rotation.

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* *Cf.* Manning—loc. cit. §. 111.

CHAPTER III.

COMPLEXES* IN n-DIMENSIONS.

38. In my Thesis on the Geometry of Hyper-Spaces (Vol. I) I had occasion to notice that a space of r-dimensions in a higher space of n-dimensions can generally be defined by $n-r$ linear equations between n variables. In the present chapter we shall study the geometrical significance of r equations connecting the n variables, in which the equations are not necessarily all linear, but one or more of them can be of any degree i.

Definition : A *Complex* is defined to be the aggregate of the system of values of the variables or the aggregate of points which satisfy a number of given conditions.

The *order of a Complex* is the number of conditions which it satisfies. Thus a Complex of order i is one which satisfies 'i' conditions.

If there are two Complexes of orders i and i', their intersection is a Complex of order $i+i'$, which is defined by the equations defining the two given Complexes. If $i+i'=n$, the intersection reduces to a finite number of points.

The *degree of a Complex of the first order* is the degree of the equation which defines the Complex.

The *degree of a Complex of order i* is the number of its intersections with a space of i dimensions which is defined by $n-i$ linear equations, *i.e.* with a Complex of the first degree and of order $n-i$.

* Bertini names it "Varieta."

If $i=n$, the degree of the Complex of order n is the number of points which define the Complex.

39. A Complex is said to be "*reducible*," when its points can be divided into two or more groups defining other Complexes, otherwise the Complex is said to be "*irreducible*."

A Complex of order i and degree r is denoted by V_i^r, which reduces to a group of r points when $i=0$.

When any space S_{n-i} of i dimensions has in common with a V_i^r, $(r+1)$ points, it has an infinite number.

We have the following general theorem :—" Any space S_k of $n-k$ dimensions $(k \geq n-i)$ intersects a V_i^r in a V^r_{k+i-n}. Any S_k which has in common with V_i^r any V^r_{k+i-n} and a point, intersects V_i^r in a Complex of dimensions $\geq k+i-n+1$ or in more Complexes of at least one dimension less."

A Complex of order i can be represented by

$$f_k(x_1, x_2, \ldots x_n)=0, \quad k=1, 2, 3, \ldots i.$$

40. The complete intersection of two Complexes can be composed of several distinct Complexes of lower orders. In this case a Complex of order i cannot be represented by only r equations between the n variables, but if there is a Complex which is the complete intersection of i Complexes of first order, it is evident that the degree of this Complex is equal to the product of the degrees of those Complexes.

Let U_1^r and V_1^r be two Complexes of order 1 and degree r, *i.e.*, let U_1^r and V_1^r be two polynoms of degree r in n variables. Then $U_1^r - V_1^r = 0$ represents a Complex of order 1 and degree r, which is the intersection of U_1^r and

$V_1^{\,r}$. If now $U_1^{\,r}$ contains a linear factor L'_1 (say), it becomes of the form $L'_1 \cdot P_1^{\,r-1} - V_1^{\,r} = 0$ (1), where $P_1^{\,r-1}$ is of degree $r-1$. But a point on a $V_i^{\,r}$ is a multiple point of order s, if any space S_{n-i} of i dimensions drawn through the point intersects it in s coincident points there, *i.e.*, if the equation for determining the intersections of S_{n-i} and $V_i^{\,r}$ has s equal roots. Now if the point at infinity on the x_nth axis is a multiple point of order $r-1$ on the Complex *i.e.* if Ox_n meets the Complex in $(r-1)$ coincident points at infinity, the equation (1) takes the form—

$x_n \cdot U_1^{\,r-1} - V_1^{\,r} = 0$, or $x_n U - V = 0$, where U and V are polynoms of degrees $r-1$ and r respectively and are independent of x_n. In this case all lines parallel to the axis Ox_n meets the Complex in only one finite point and it is called a Monoid.*

41. Consider a Complex of the first order in n variables—

$$f(x_1, x_2 \ldots x_n) = 0.$$

Let $x'_1, x'_2, x'_3 \ldots x'_n$ be the co-ordinates of any point P. Through this point we can draw $(n-2)$ Complexes of the first order and first degree, *i.e.* we can draw a plane passing through this point and parallel to the plane of (x_1, x_2) :—

$$\left.\begin{aligned} x_3 - x'_3 &= A_3(x_1 - x'_1) + B_3(x_2 - x'_2) \\ x_4 - x'_4 &= A_4(x_1 - x'_1) + B_4(x_2 - x'_2) \\ \ldots\quad & \quad\ldots \qquad\qquad \ldots \\ x_n - x'_n &= A_n(x_1 - x'_1) + B_n(x_2 - x'_2) \end{aligned}\right\} \quad \ldots \quad (1)$$

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* This is an extension of the term "monoid" used by M. Cayley—Comptes rendus Vols. LIV and LVIII.—Cayley says that if a surface contain a multiple point of order less by one than its degree, the lines drawn through this point intersect the surfaces only in one point and the surface is called a *Monoid*.

If we substitute the values of x_3, $x_4 \ldots x_n$ from this equation in $f(x_1, x_2 \ldots x_n)=0$, we obtain a relation of the form $F(x_1, x_2)=0$; but when $x_1=x'_1$, $x_2=x'_2$, the equations (1) give $x_3=x'_3$, $x_4=x'_4, \ldots x_n=x'_n$. Therefore $F(x'_1, x'_2)$ is the value of $f(x_1, x_2 \ldots x_n)$ when $x'_1, x'_2 \ldots x'_n$ are put for $x_1, x_2 \ldots x_n$ respectively, *i.e.*,

$$F(x'_1, x'_2)=f(x'_1, x'_2, x'_3, \ldots x'_n).$$

Thus the above plane intersects the Complex in the plane curve $F(x_1, x_2)=0$.

Let us take another Complex $\phi(x_1, x_2, \ldots x_n)=0$. Suppose that the intersection of f and ϕ is composed of several distinct Complexes of second order. Let a plane intersect these two Complexes. Then we can determine as above in any of them a plane curve. The intersection of these two plane curves $F(x_1, x_2)=0$ and $F_1(x_1, x_2)=0$ is the intersection of these Complexes $f=0$ and $\phi=0$ with the plane, *i.e.* with the $(n-2)$ Complexes of the first degree. If the Complexes f and ϕ are reducible, their intersections are also reducible into several groups defined by different equations. Let p be the number of points defining one of these groups (supposed irreducible). The number of points which coincide at any one of these points is equal to the sum of the orders of $F(x_1, x_2)=0$, the point (x_1, x_2) being placed successively on the different branches of $\phi=0$, very close to the point.*

Now since $F(x_1, x_2)$ is nothing but $f(x_1, x_2, \ldots x_n)$, we easily obtain the theorem :—

If the intersection of two Complexes of the first order f and ϕ consists of distinct Complexes of second order, the point $(x_1, x_2 \dots x_n)$ may be placed successively on the different branches of ϕ, at a very small distance from any of the points, such that the sum of the orders of the quantities $f(x_1, c_2 \dots x_n)$, multiplied by the degree of the Complex of the second order, is equal to the number of intersections of f and ϕ.

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* *Vide* Bull. de la Soc. Math. de France, Vol- 2. Theorem II, p. 35.

42. If a Complex V_i^r of order i and degree r is irreducible, the r points in which any space S_{n-i} of i dimensions intersects it are all distinct. Suppose the r points in which any space S_{n-i} intersects V_i^r are divided into two groups :—one containing α points each counted twice and another of β points ($\therefore 2\alpha+\beta=r, \alpha>0, \beta\geq 0$). It follows therefore that the equation of the nth degree which determines the r points has α double roots and β simple roots. These roots may separately be obtained by two different equations containing the co-efficients of the given equations. It follows then that V_i^r consists of two parts defined by the equations of V_i^r and the two equations giving the roots, which cannot be. Thus the points must all be distinct.

We conclude therefore that every irreducible Complex V_i^r is intersected by any space S_{n-i+1} in an irreducible V_1^r. If not, suppose that the section of V^r_i by S_{n-i+1} consists of more curves $V_1^{r_1}$, $V_1^{r_2}$, $\dots V_1^{r_t}$,

$$(r_1 + r_2 + \dots + r_t = n).$$

By varying the S_{n-i+1} in a fixed space S_{n-i}, it will intersect V_i^r in r distinct points, of which r_1 will be on $V_1^{r_1}$, r_2 on $V_1^{r_2}$, $\dots r_t$ on $V_1^{r_t}$. These t curves will describe t distinct Complexes. The Complex V^r_i is then composed of these t Complexes (represented by the equations defining the t curves).

From this we deduce the more general theorem :— Every irreducible Complex V_i^r is intersected by an $S_{n-i+t}(t>0)$ in an irreducible V_t^r.*

43.† Let us now consider the representation of a Complex of any order higher than one. For the points at infinity in a Complex of order i, the ratios of the Coordinates $x_1, x_2, \ldots x_n$ to one of them are connected by i relations. The ratios $\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots \frac{x_{n-1}}{x_n}$ may be regarded as $(n-1)$ new co-ordinates $\xi_1, \xi_2, \ldots \xi_{n-1}$ and therefore the i relations define a Complex of order i in $(n-1)$ dimensions. Now, one or more of the new co-ordinates can be zero or infinite, but this particular case can easily be considered by a simple linear transformation.

Let $A=0$ and $B=0$ be two Complexes of the first order. Eliminating x_n between these two equations we obtain an equation of the form $\Delta=0$, where Δ does not Contain x_n. The Complex of the second order $A=0$, $B=0$ is reducible or not, according as Δ does or does not contain factors. Let $f_1(x_1, x_2, \ldots x_{n-1})$ be an irreducible factor of Δ. Then the equations $f_1(x_1, x_2, \ldots x_{n-1})=0$, and $vx_n-u=0$...(1) define an irreducible Complex of second order. It is evident that each system of values of $x_1, x_2, \ldots x_{n-1}$ satisfying $f_1=0$ gives only one value of x_n from the second equation. Thus we cannot say that the Complex is the complete intersection of the two Complexes (1), but its intersection consists of the two Complexes $f_1=0$ and $v=0$.

Suppose that the Complex is irreducible and that for infinite values of the co-ordinates of its points, the ratios

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* Cf. Bertini—Introduzioni alla Geometria etc. Cap 9°. n.4.

† The monoidal representation of Complex was given by Bertini in 1907 -See loc. cit. Chap. 9. §. 28.

This method of representation was also given by M. Halphen in 1875 in the Bull. de la Soc. Math. de France, Vol. II.

$\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots \frac{x_{n-1}}{x_1}$ are neither of them zero or infinite. Let the sum of higher degree terms in f_1 be f_2, so that by equating f_2 to zero we obtain a relation between the ratios $\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots \frac{x_{n-1}}{x_1}$. Thus $f_1 = f_2 + f_3$, where f_3 is of degree less than f_1. If the higher degree terms in v contain the factor f_2, we have $v = \mathrm{P}\cdot f_2 + v_1$, v_1 being of degree less than v. Thus for representing the Complex we may replace v by $v - \mathrm{P}f_1 = v_1 - \mathrm{P}f_3$. If $v_1 - \mathrm{P}f_3$ still contains the factor f_2 in its higher degree terms, we may apply the same transformation to the new expression, and repeat the same process until we obtain an expression which does not contain f_2 in its higher degree terms. Similarly, u may be transformed. Thus u and v may be supposed not to contain the factor f_2 in their higher degree terms. Therefore, for infinite values of the co-ordinates $x_1, x_2, \ldots x_{n-1}$ which make $f_2 = 0$, u and v are infinite of an order denoted by their degrees. Consequently x_n must be of the same order as the other coordinates.

Next, it is evident that x_n cannot be infinite for the values of the other co-ordinates which make $v = 0$ and $f_2 = 0$, because the Complex of the points at infinity is irreducible. Therefore the ratio $\frac{u}{v}$ is finite, which shows that u passes through the intersection of $v = 0$ and $f_2 = 0$ and u is small of the same order as v, while $x_1, x_2, \ldots x_{n-1}$ is very near to the intersection on a branch of f_1. Therefore, from what has been said before, it follows that the intersection of v and f_1 which lies on u counts as so many intersections of u and v as of v and of f_1. Thus, if n is the degree of f_1, the intersection counts as $(p-1)n$, where $p-1$ is the degree of v and p that of u. The remaining intersection of u and f_1 is $pn - (p-1)n = n$.

Now we apply the transformation—

$$x'_1=\frac{1}{x_1},\ x'_2=\frac{x_2}{x_1},\ x'_3=\frac{x_3}{x_1},\ldots\ldots\ldots x'_n=\frac{x_n}{x_1}.$$

In the new Complex, the part at infinity is irreducible. The Complex is consequently represented by the equations $v'x'_n-u'=0, f'_1=0$, where u' and v' are respectively of degrees p and $p-1$. In this case, as also referred to the old system, the Complex is represented by $vx_n-u=0$ and $f_1=0$, where u and v are of the same degrees as u' and v'. Therefore, as before, the intersection of u and f_1 is the same as that of v and f_1, and the intersection consists of $(p-1)n$ units and a Complex of degree n. The same conclusion holds for reducible Complexes.

Now, since u passes through all the intersections of v and f_1 and this intersection counts as as much units in the degree of $u=0, f_1=0$ as in that of $v=0$, $f_1=0$, it follows that if no part of this intersection is contained in f_1, u is of the form $lv+mf_1$, where l is of the first degree. The monoid $vx_n-u=0$ is of the form $x_n=l$. Therefore the Complex of the second order is traced on a Complex of first order and first degree.

In a similar way, we can easily show that all Complexes of the ith order are represented by an equation $\psi=0$ between $n-i+1$ co-ordinates $x_1, x_2, x_3, \ldots x_{n-i+1}$, and $i-1$ monoids $x_{n-i+1}=\frac{u}{v}$, $x_{n-i+2}=\frac{u'}{v'}$, ……satisfying the same conditions as above. We may reduce the equations of these monoids to the same denominator v; thus

$$x_{n-i+2}=\frac{u_{n-i+2}}{v},\ x_{n-i+3}=\frac{u_{n-i+3}}{v},\ \ldots\ldots x_n=\frac{u_n}{v}.$$

Each of these Complexes $u_{n-i+2}, u_{n-i+3}, \ldots\ldots$ is of a degree higher by one than that of v, and passes through the intersections of v and ψ. The degree of such a Complex is precisely that of ψ.

44. The degree of the intersection of two Complexes, of which one is of the first order, is equal to the product of the degrees of these Complexes.

Let us consider a Complex of order i and degree r given by the equations :—

$$\phi=0,\ x_{n-i+2}=\frac{u_{n-i+2}}{v},\quad \ldots\ldots\quad x_n=\frac{u_n}{v}. \qquad \ldots \quad (1)$$

Let A be a Complex of the first order given by—

$$A=a+a_1x_1+a_2x_2+\ \ldots\ldots\ +a_nx_n=0 \qquad \ldots \quad (2)$$

Substituting the values of $x_{n-i+2}, \ldots x_n$ in (2) we obtain

$$\psi=(a+a_1x_1+\ldots+a_{n-i+1}x_{n-i+1})v+a_{n-i+2}u_{n-i+2}+\ldots$$
$$+a_nu_n=0. \qquad \ldots \quad (3)$$

It follows therefore that if the degree of the Complex C is r, that of ψ is kr, where k is the degree of u.

If we put $v=0$, then $a_{n-i+2}u_{n-i+2}+\ldots+a_nu_n=0$. Therefore $v=0$, $\phi=0$ gives a Complex which counts for $(k-1)mr$ of the intersections of $\psi=0$, $\phi=0$. The remaining intersection is of degree mr (where m is the degree of ϕ) and it is the degree of the intersection of the two Complexes $v_i{}^r$ and A.

Cor :—When we consider a three-dimensional space, the above theorem expresses the fact that the number of intersections of a curve and a surface is equal to the product of the degrees of this curve and the surface.*

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* Salmon—Geometry of three dimensions, Ch. XII, §. 331.

45. The above theorem can be generalised as follows:—

The degree of the intersection of two Complexes, of which the sum of the orders does not exceed the number of dimensions, is equal to the product of degrees of these Complexes.

Let V_i^r and $W_{i'}^{r'}$ be two Complexes of orders i and i' and degrees r and r' respectively, defined by equations analogous to those given in the preceding article.

Let us transform the equation $V_i{}^r$ to a space of $n+i-1$ dimensions. Thus we obtain the equations of a Complex $C_i{}^r$ in $n+i-1$ dimensions in the form

$$\psi(x_1, x_2, \ldots x_n)=0,\ \xi_1=\frac{u_1}{v}, \xi_2=\frac{u_2}{v}.\ \ldots\ldots \xi_{i-1}=\frac{u_{i-1}}{v}.$$

Similarly $W_{i'}^{r'}$ is transformed to a space of $n+i'-1$ dimensions and the Complex $C'^{r'}_{i'}$ is given by

$$\psi'(x_1, x_2, \ldots x_n)=0,\ \xi'_1=\frac{u'_1}{v'},\ \xi'_2=\frac{u_2}{v'}, \ldots \xi'_{i'-1}=\frac{u_{i'-1}}{v'}.$$

The degrees of ψ, ψ', $u_1, \ldots u'_1 \ldots$ are determined as before. Thus the two Complexes C_i^r, $C_{i'}^{r'}$ may be considered as both belonging to the same space of $n+i+i'-2$ dimensions, the variables being denoted by (say) $x_1, x_2, x_3, \cdots x_n, \xi_1, \xi_2, \ldots \xi_{i-1}, \xi'_1, \xi'_2, \ldots \xi'_{i'-1}$.

Now we can easily obtain the intersection of these two Complexes by eliminating one of the old co-ordinates $x_1, x_2 \ldots x_n$ (say x_n) between the equations $\psi=0$ and $\psi'=0$, which gives an equation $\phi=0$ of degree rr' in $(x_1, x_2, \ldots x_{n-1})$, and then deducing from the equations an equation of the form $vx_n-u=0$. Then $\phi=0$, $vx_n-u=0$, together with the equations for $\xi_1, \xi_2, \ldots \xi'_1, \xi'_2, \ldots$ obtained from the scheme of transformation, will give the required intersection. These also give us a Complex$\left(C^{rr'}_{i+i'} \right)$ of order $i+i'$ in $n+i+i'-2$ dimensions of degree rr'.

The intersection of this Complex $\left(C^{rr'}_{i+i'} \right)$ with the space or Complex $\xi_1=0, \xi_2=0, \ldots \xi_{i-1}=0, \xi_1'=0, \xi_{2\,1}'=0, \ldots \xi'_{i-1}=0$, is also a Complex of the same order rr'. This is really the intersection of the two given Complexes V^r_i, $W^{r'}_{i'}$, which proves the theorem.

46. We shall next try to give a geometric interpretation to the equations defining a Complex of order i and degree r in n dimensions. In the first place let us consider a Complex in a three-dimensional space. In the above monoidal representation, a curve in space can be represented by—

$$f(x, y)=0, \qquad z\phi_1=\phi_2,$$

where ϕ_1 and ϕ_2 are polynoms of orders $p-1$ and p respectively, and f is the equation of a cone (cylinder) of order m. The two surfaces f and ϕ_1 intersect in $m(p-1)$ lines, since $\phi_1=0$ is tangent to the monoid at the multiple point of order $p-1$.

If we draw k chords of the gauche curve through any point of space, they are double lines of the cone $f=0$, and are counted twice among the lines $m(p-1)$. Denoting by k' the remaining lines we have

$$m(p-1)=2k+k'.$$

Also the $k+k'$ lines are simple lines of the monoid passing through the multiple point, whence we must have*

$$p(p-1)\geq k+k'$$

Eliminating k' we obtain—

$$\frac{m(p-1)}{2}\geq k\geq (p-1)(m-p)$$

But certainly p is greater than unity.

$$\therefore \frac{m}{2}\geq (m-p)$$

Therefore the inferior limit to the order p of the monoid is

$$p\geq \frac{m}{2}\cdot$$

Thus we may regard a curve in space as a curve traced on the surface of a cylinder.

If now we Consider a Complex of order $n-1$ in n dimensions, which is represented by

$$f(x_1, x_2)=0,\ x_3=\frac{u_3}{v},\ x_4=\frac{u_4}{v},\ldots\ldots x_n=\frac{u_n}{v}$$

we see that each group of the equations

$$f(x_1, x_2)=0,\ \ x_i=\frac{u_i}{v}\ [i=1, 2, 3, 4, \ldots\ldots n]$$

represents a curve in space. We can therefore say that a Complex of order $n-1$ is represented by $(n-2)$ gauche curves traced upon the same cylinder.

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* *Vide*—Bertini—loc. cit. Cap. 8°, *n*.15.

47. We shall now conclude this chapter by stating a most important theorem* in the theory of Complexes in n-space:—

"A Complex V_k^r of order r and k dimensions and a Complex $V_{n-k}^{r'}$ of order r' and $(n-k)$ dimensions, which have not infinite number of common points, intersect in rr' points.

The theorem is easily verified in the case when the Complex V_k^r (for example) is composed of r spaces S_k.

It follows that two Complexes V_k^r and $V_{k'}^{r'}$, if $k+k'>n$, intersect in a Complex $V_{k+k'-n}^{rr'}$, if they have not in common a Complex of dimension greater than $k+k'-n$.

In fact, any space $S_{2n-k-k'}$ in n-space intersects a Complex $V_{k'}^{r'}$ (for example) in a Complex $V_{n-k}^{r'}$ which has, by the preceding theorem, rr' common points (not infinite) with the Complex V_k^r, as otherwise varying the space $S_{2n-k-k'}$ we obtain a Complex common between V_k^r and $V_{k'}^{r'}$, of dimensions $>n-k-k'$.

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* Halphen has given a demonstration of this theorem in the monoidal representation of Complexes (Bull. de la Soc. Math. de France, Vol. 2, 1874, p. 34). But it was more rigorously and more simply proved by Noether (Math. Ann. Vol, 11, 1877, p. 570). Another proof was given by Pieri (Giornale di Matematiche, Vol. 26(1), 1888).

When V^{r}_{k} , $V^{r'}_{k'}$ have a common Complex of dimensions $> k + k' - n (\geq 0)$, the two Complexes may have another common Complex, but the dimension of any of these is always $\geq k + k' - n$.

48. Geometrical Continuum : Each group of values of the variables satisfying the equations of a Complex is a solution. If the number of equations be exactly equal to the number of variables the solution is determinate, but if less, the aggregate of values is defined to be a "Continuum." If there are i variables, then i represents the dimension of the continuum ; when all the equations of a Complex are linear, the continuum is linear. The continuum defined by one equation between n variables is called an $(n-1)$ continuum, as we have already seen.

The aggregate of values of the variables even if they are not connected by any equation is called a "region," more precisely n-region. If the variables are independent, but the boundary defined by the n-tuple Integral $\int dx_1 dx_2 dx_3 \ldots$, in which none of the variables are infinite, then the aggregate of values over which the integral extends is called a closed region and the integral represents its *content* or *mass*.

If $x_1, x_2, \ldots$ are the variables, $dx_1, dx_2, \ldots$ their differentials, supposing all independent,

then $s = \int \sqrt{dx_1^2 + dx_2 + \ldots}$ represents the length of the distance A to B, where A and B are the limits of the integral. The calculus of variation shows that this distance is a minimum if the variable functions are of the first degree.

49. Application of the orthogonal transformation : We have already discussed the properties of an orthogonal transformation. Let $x_1, x_2, \ldots$ be the original variables and $t_1, t_2, \ldots$ the new variables.

Let
$$\left.\begin{aligned} x_1 &= \alpha_1 t_1 + \alpha_2 t_2 + \ldots \\ x_2 &= \beta_1 t_1 + \beta_2 t_2 + \cdots \\ &\text{etc.} \qquad \text{etc.} \end{aligned}\right\} \quad \ldots \quad (1)$$

Then,
$$r^2 = x_1^2 + x_2^2 + x_3^2 + \ldots \quad = (\alpha_1^2 + \ldots) t_1^2 + \text{etc.}$$
$$+ 2(\alpha_1 \alpha_2 + \beta_1 \beta_2 + \ldots) t_1 t_2 + \ldots \text{etc.}$$

where $r =$ distance of the point from the common origin. If $r^2 = t_1^2 + t_2^2 + \ldots$etc., then the constants of transformation must satisfy the conditions, as we have already seen,

$$\left.\begin{aligned} \alpha_1^2 + \beta_1^2 + \ldots \quad &= 1, \text{ etc.} \\ \alpha_1 \alpha_2 + \beta_1 \beta_2 + \ldots &= 0, \text{ etc.} \end{aligned}\right\}$$

If
$$\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots \\ \beta_1 & \beta_2 & \beta_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

then,
$$\Delta^2 = \begin{vmatrix} \Sigma\alpha_1^2 & \Sigma\alpha_1\alpha_2 & \Sigma\alpha_1\alpha_3 & \ldots \\ \Sigma\alpha_2\alpha_1 & \Sigma\alpha_2^2 & \Sigma\alpha_2\alpha_3 & \ldots \\ \Sigma\alpha_3\alpha_1 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

Thus, by the above conditions, $\Delta^2 = 1$ and consequently

either $\Delta = -1$ or $\Delta = +1$.

If $\Delta = -1$, the new variables are simply the old ones with a negative sign. $\therefore \Delta = +1$.

If now $a_1, b_1, c_1, \ldots$ are the complements of $\alpha_1, \beta_1, \gamma_1 \ldots$

i.e. if $\quad a_1 = \dfrac{\partial \Delta}{\partial \alpha_1}, \quad b_1 = \dfrac{\partial \Delta}{\partial \beta_1}, \quad c_1 = \dfrac{\partial \Delta}{\partial \gamma_1}$, etc.

then $\quad \Delta t_1 = a_1 x_1 + b_1 x_2 + c_1 x_3 + \ldots$etc.

But if we multiply the transformation formulae respectively by $\alpha_1, \beta_1, \gamma_1, \ldots$ and add, then by condition (1),

$$t_1 = \alpha_1 x_1 + \beta_1 x_2 + \ldots$$

Thus, if $\Delta = 1$, $a_1 = \alpha_1$, $b_1 = \beta_1$, etc. *i.e.* the complements are equal to the corresponding elements themselves.

Hence $\quad a_1 \alpha_1 + a_2 \alpha_2 + \ldots = \Delta$, etc.

$\quad a_1 \beta_1 + a_1 \beta_2 + \ldots = 0$, etc.

Therefore, $\quad \alpha_1{}^2 + \alpha_2{}^2 + \alpha_3{}^2 + \ldots = 1$, etc.

$\quad \alpha_1 \beta_1 + \alpha_2 \beta_2 + \ldots = 0$, etc.

Hence we see that we can transform the original variables to new ones and vice versa and the processes are similar.

50.* The content of the cylinder: The mass V of an open region is represented by the n-tuple integral $\int^n dx_1 dx_2 \ldots$. If now the $n-1$ tuple integral $\int^{n-1} dx_2 dx_3 \ldots$ has a constant value A independent of x_1, and if the limits are constant with respect to x_1, and their difference is a, then $V = a$A.

The first condition among others is satisfied, if the boundary equation be given in the form

$$\phi(x_2 - p, x_3 - q, \ldots) = 0,$$

where $p, q, \ldots$ are functions of a single variable x_1.

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* Cf. L. Schlaffli—Theorie der vielfachen Kontinuität, edited by J. H. Graf, Bern, 1901—§. 5, p. 11.

There are now two boundary equations of the form x_1=constant, and the integral V extends over all values of x_1 which lie between those two constants. If however $p, q, \ldots$ are linear functions of x_1, then the boundary defined by $\phi=0$ is generated by the motion of a line which remains always parallel to those determined by $x_2=p, x_3=q, \ldots$ The closed region V is then the cylinder of ordinary Geometry, where A is the base and a is the corresponding height. We may therefore state the theorem in the following form :—

The content of cylinder is equal to the product of its base and height.

51. The Paralleloschem :* If the boundary of the $(n-1)$-tuple Integral A (the base) in the preceding article is again similarly determined, and so on, we have $V=abc\ldots$ Then x_1 lies between the two constants whose difference is a, x_2 lies between two linear functions of x_1 whose difference is b, x_3 between two linear functions of x_1, x_2, whose difference is c, and so on. The region is thus enclosed between n pairs of parallel linear continuum, and we call it the "Paralleloschem." We always assume that the n original equations are all satisfied by the zero values of the variables. The $(n-1)$ of these original linear equations taken together will determine a straight line, bounded by the remaining pair of parallel linear continuum and this line is called an "edge" of the paralleloschem. There are in all $n.2_{n-1}$ edges ; but 2^{n-1} of them are parallel and of equal length. Thus they fall into n groups, of which only n pass through the origin. The first is $x_1=0$, the second is $\alpha_1 x_1+\beta_1 x_2=0$, the third is $\alpha_2 x_1+\beta_2 x_2+\gamma_2 x_3=0$, and so on. If we take the first, no variable vanishes; if the second, $x_1=0$; if the third, $x_1=0, x_2=0$; if the fourth, $x_1=x_2=x_3=0$; and so on. For the first edge, no projection vanishes, and its first projection is a, for the second edge the first projection is zero and the second is b ; and so on.

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* The name is used by Schlafli.

52. The content or mass of the Paralleloschem : The mass or content of a "Paralleloschem" is equal to the determinant of the orthogonal projections of its edges. Let the projections of the edge of a paralleloschem in any rectangular system be $a_1, b_1, c_1, \ldots$; $a_2, b_2, c_2, \ldots$ $a_3, b_3, c_3, \ldots$ We now transform this to a new system to which the paralleloschem has the above relation. If now we consider the edges as the new variables X, Y.... in any of the above-named order, then the projections in the new system are :

$$\begin{array}{l} A_1,\ 0,\ 0,\ 0, \ldots \\ A_2,\ B_2,\ 0,\ 0, \ldots \\ A_3,\ B_3,\ C_3,\ 0, \ldots \\ \ldots \quad \ldots \quad \text{etc.} \end{array}$$

Let

$$\begin{array}{l} x_1 = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 - \ldots \\ x_2 = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \ldots \\ \quad \text{etc.} \qquad \text{etc.} \qquad \text{etc.} \end{array}$$

then we have

$$(1)\ \left.\begin{array}{l} a_1 = A_1 \alpha_1 \\ b_1 = A_1 \beta_1 \\ \ldots \quad \ldots \end{array}\right\} \qquad (2)\ \left.\begin{array}{l} a_2 = A_2 \alpha_1 + B_2 \alpha_2 \\ b_2 = A_2 \beta_1 + B_2 \beta_2 \\ \ldots \quad \ldots \end{array}\right\}$$

$$(3)\ \left.\begin{array}{l} a_3 = A_3 \alpha_1 + B_3 \alpha_2 + C_3 \alpha_3 \\ b_3 = A_3 \beta_1 + B_3 \beta_2 + C_3 \beta_3 \\ \ldots \quad \ldots \quad \ldots \end{array}\right\}$$

From the first of these equations we have

$$A_1 = \sqrt{a_1^2 + b_1^2 + \ldots}, \text{ and } \alpha_1 = \frac{a_1}{A_1}, \ \beta_1 = \frac{b_1}{A_1}, \ldots$$

Since the second system is also orthogonal, from the second we have

$$A_2 = a_2\alpha_1 + b_2\beta_1 + \ldots$$

and if we substitute the above value of A_1, we have

$$B_2 = \sqrt{(a_2 - A_2\alpha_1)^2 + (b_2 - A_2\beta_1)^2 + \ldots} \text{ etc.}$$

$$\alpha_2 = \frac{a_2 - A_2\alpha_1}{B_2}, \ldots \text{ etc.}$$

The third equations give—

$$A_3 = a_3\alpha_1 + b_3\beta_1 + \ldots, \; B_3 = a_3\alpha_2 + b_3\beta_2 + \ldots$$

and from these we finally obtain $C_3, \alpha_3, \beta_3 \ldots$ and so on. Each system of values contained in the paralleloschem is represented by the equations—

$$\begin{aligned} x_1 &= \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \ldots \\ x_2 &= \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \ldots \\ &\text{etc.} \quad \text{etc.} \quad \text{etc.} \end{aligned}$$

where the indeterminate co-efficients are positive and fractional. If the determinant $V = \Sigma \pm a_1 b_2 c_3 \ldots$ be multiplied by itself, then the product is a determinant whose elements are

$$\begin{matrix} a_1^2 + b_1^2 + c_1^2 + \ldots & a_1a_2 + b_1b_2 + c_1c_2 + \ldots \\ a_2a_1 + b_2b_1 + c_2c_1 + \ldots & a_2^2 + b_2^2 + c_2^2 + \ldots \\ \text{etc.} & \text{etc.} \\ & a_1a_3 + b_1b_3 + c_1c_3 + \ldots \text{etc.} \\ & a_2a_3 + b_2b_3 + c_2c_3 + \ldots \text{etc.} \\ & \text{etc} \end{matrix}$$

If now we denote the edges by $k_1, k_2, k_3, \ldots$ and the cosine of the angle included between two edges as $(k_1 k_2)$, then

$$a_1{}^2+b_1{}^2+c_1{}^2+\ldots=k_1{}^2,\ a_1a_2+b_1b_2+\ldots=k_1k_2(k_1k_2)$$

and so on.

$$\text{Then}\quad V^2=\begin{vmatrix} k_1{}^2 & k_1k_2(k_1k_2), & k_1k_3(k_1k_3)\ldots & \\ k_2k_1(k_2k_1) & k_2{}^2 & k_2k_3(k_2k_3)\ldots & \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$=(k_1k_2k_3\ldots)^2\times\begin{vmatrix} 1 & (k_1k_2) & (k_1k_3) & \ldots \\ (k_2k_1) & 1 & (k_2k_3) & \ldots \\ (k_3k_1) & (k_3k_2) & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

Hence, the mass or content of the Paralleloschem is the product of its edges multiplied by the square root of a determinant whose general terms are the cosines of the angles included between the edges in pairs.

This method we have already discussed in Geometry of Hyper-spaces Part I.

If $V=0$, then the edges all satisfy the same linear equation and they lie in one and the same plane, and the above determinant also vanishes.

If $n=4$, we obtain the same result as before in Chapter II. *i.e.*

$$\begin{vmatrix} 1 & \cos a_1 & \cos b_1 & \cos c_1 \\ \cos a_1 & 1 & \cos c_2 & \cos b_2 \\ \cos b_1 & \cos c_2 & 1 & \cos a_2 \\ \cos c_1 & \cos b_2 & \cos a_2 & 1 \end{vmatrix}$$

$$=1-\Sigma\cos^2 a_1+2\Sigma\cos a_2\cos b_2\cos c_2+\Sigma\cos^2 a_1\cos^2 a_2$$
$$-2\Sigma\cos a_1\cos a_2\cos b_1\cos b_2.$$

This is the case with a tetrahedron.

53. Mass or Content of the Pyramide :—

If the Integral $P = \triangle \int^{n} dt_1 dt_2 dt_3, \ldots$ is bounded by the conditions $t_1 > 0,\ t_2 > 0$, etc.

$$\frac{t_1}{k_1} + \frac{t_2}{k_2} + \frac{t_3}{k_3} + \ldots < 1,$$

we call such an Integral P of $n+1$ linear continuum a Pyramide.

If we now put $t_1 = k_1 u_1,\ \ t_2 = k_2 u_2,\ \ t_3 = k_3 u_3 \ldots$

then $$P = \triangle \cdot k_1 k_2 k_3 \ldots \times \int^{n} du_1 du_2 du_3 \ldots$$

with the conditions $u_1 > 0,\ u_2 > 0,\ u_3 > 0, \ldots$etc.

$$u_1 + u_2 + u_3 + \ldots < 1.$$

Since the integral contains no constant, it can be represented by $f(n)$.

$$\text{In} \int^{n-1} du_2 du_3 du_4 \ldots [u_2 > 0, u_3 > 0, \ldots, u_2 + u_3 + u_4 + \ldots < 1 - u]$$

let us put $u_2 = (1 - u_1) v_2,\ u_3 = (1 - u_1) v_3, \ldots$etc. then

$$\int^{n-1} du_2 du_3 \ldots = (1 - u_1)^{n-1} \int^{n-1} dv_2 dv_3 dv_4 \ldots$$

$$[v_2 > 0,\ v_3 > 0,\ \ldots\ v_2 + v_3 + v_4 + \ldots < 1]$$

$$= (1 - u_1)^{n-1} f(n-1).$$

$$\therefore f(n) = f = (n-1) \cdot \int_0^1 (1 - u_1)^{n-1} du_1 = \frac{f(n-1)}{n} = \frac{1}{1 \cdot 2 \cdot 3 \ldots n},$$

for, $$f(1)=\int du_1[u_1>0,\ u_1<1]=1$$

$$\therefore\ P=\frac{\Delta\cdot k_1k_2k_3\ldots}{1\cdot2\cdot3\ldots n}=\frac{V}{1\cdot2\cdot3\ldots n}.$$

∴ The content of the pyramide is equal to the content of the paralleloschem which has n edges common, divided by $n!$.

CHAPTER IV.

HYPER-SURFACES.

54. We shall now introduce the idea of Homogeneous Coordinates in the Geometry of Hyper-spaces. As in the case of plane and solid Geometries, the introduction of Homogeneous co-ordinates in the geometry of hyper-spaces greatly simplifies the study of projective properties of curves and hyper-surfaces. Prof. E. Bertini* of the University of Pisa has written a Treatise on the "Introduction to the projective Geometry of Hyper-spaces," in which he dwells at some length on the projective properties of curves and surfaces.

In studying properties of Hyper-surfaces we shall make use of homogeneous coordinates, but at the outset we shall show that some geometrical interpretations can be given to this special system of co-ordinates.

55. We have so far defined the position of a point with reference to n mutually orthogonal axes, called "the axes of Coordinates." The position of a point may also be defined with reference to a "Simplicissima," determined by $(n+1)$ given independent† points in the general n-space. The $(n+1)$ vertices will be called the "fundamental points." The $(n+1)$ co-ordinates of a point will be defined as the ratios of the n-dimensional contents of the joins,‡ which have the given point and any n of the $(n+1)$ fundamental points as vertices, to the content of the fundamental Simplicissima.

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* Pisa, December, 1906.

† *Vide*—Geometry of Hyper-spaces, Part I, §. 7, published by the University of Calcutta, 1917.

‡ Bull. of the Cal. Math. Soc., 1909, Vol. 3. "on Parametric Co-efficients, etc.

Let us denote the fundamental points by $A_0, A_1, A_2, \ldots A_n$. Let u_r denote the $(n-1)$ dimensional content of the join of n of the fundamental points with the exception of A_r. Thus we obtain $u_0, u_1, \ldots u_n$ similar $(n-1)$-dimensional contents.

Let P be any given point in the n-space. Let $V_0, V_1, V_2, \ldots V_n$ denote the n-dimensional contents having P as vertex and $u_0, u_1, u_2, \ldots u_n$ as bases respectively.

Then, $V_0/\triangle, V_1/\triangle, \ldots V_n/\triangle$, where $\triangle$ denotes the content of the fundamental Simplicissima, are defined to be the $(n+1)$ co-ordinates of the point P, corresponding to the Areal or Four-plane co-ordinates in ordinary two or three dimensional geometries respectively. These $(n+1)$ co-ordinates will be denoted by the lettess $x_0, x_1, x_2, \ldots x_n$.

From the above definition it easily follows that these $(n+1)$ coordinates are connected by a linear relation

$$x_0 + x_1 + x_2 + \ldots + x_n = 1 \qquad \ldots \quad (1)$$

It is to be noticed that this relation must always hold, wherever the point is taken in the n-space. In the case when the point P lies within the join of the fundamental points, it is easily verified, having regard to the fact that the sum of $V_0, V_1, V_2, \ldots V_n$ is equal to $\triangle$. When the given point lies outside the join, regard must be had to the signs of the contents $V_0, V_1, V_2, \ldots V_n$; for a point inside the join, the signs are all positive; for a point outside, the sign is to be determined by the fact that it is regarded positive if the single perpendicular drawn from the given point on any of the bases is in the same direction as the perpendicular drawn from the corresponding fundamental point, and negative in the opposite case.

Further, it should be noticed that none of the co-ordinates of a point in this system can be infinite, and all of them cannot be simultaneously zero. The point whose coordinates are proportional to 1,1,1,...1, is called the "unit-point" and is denoted by U.

In virtue of the relation (I) all equations in this system can be made homogeneous.

56. Formulae of transformation from the Cartesian to Contental system :—

If $x_i^{(j)}$ $(j=0,1,2,\dots n\,;\ i=1,2,3,\dots n)$ be the Cartesian co-ordinates of the fundamental points and $x_1, x_2, \dots x_n$ the Cartesian co-ordinates of P referred to any system of axes, orthogonal or oblique, and if $X_0, X_1, X_2, \dots X_n$ be the contental co-ordinates of P referred to the fundamental Simplicissima, then

$$x_i = x_i^{(0)} X_0 + x_i^{(1)} X_1 + x_i^{(2)} X_2 + \dots + x_i^{(r)} X_r \dots + x_{ni} X_n \quad .. \quad \text{(I)}$$

$$(i=1,2,3,\dots n).$$

For,

$$X_0^{2*} = \frac{V_0^2}{\Delta^2} = \frac{(n!V_0)^2}{(n!\Delta)^2}$$

$$= \Sigma \begin{vmatrix} 1 & x_1 & x_2 & \dots & x_n \\ 1 & x_1^{(1)} & x_2^{(1)} & \dots & x_n^{(1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_1^{(n)} & x_2^{(n)} & \dots & x_n^{(n)} \end{vmatrix}^2$$

$$\div \Sigma \begin{vmatrix} 1 & x_1^{(0)} & x_2^{(0)} & \dots & x_n^{(0)} \\ 1 & x_1^{(1)} & x_2^{(1)} & \dots & x_n^{(1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_1^{(n)} & x_2^{(n)} & \dots & x_n^{(n)} \end{vmatrix}^2$$

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* *Vide* Proceedings of the London Math. Soc. Vols. XVIII and XXI.

$$\equiv[1xx^{(1)}\,x^{(2)}\,\ldots\,x^{(n)}]^2 \div [1x^{(0)}\,x^{(1)}\,\ldots\,x^{(n)}]^2$$

$$\therefore X_0=[1xx^{(1)}\,\ldots\,x^{(n)}] \div [1x^{(0)}\,x^{(1)}\,\ldots\,x^{(n)}]$$

or, $[1x^{(0)}\,x^{(1)}\,x^{(2)}\,\ldots\,x^{(n)}]X_0=[1x^{(1)}\,x^{(2)}\,\ldots\,x^{(n)}]$.

Similarly,

$$[1x^{(0)}\,x^{(1)}\,x^{(2)}\,\ldots x^{(n)}]X_1=[1xx^{(0)}\,x^{(2)}\,\ldots\,x^{(n)}]$$

etc. etc.

and generally,

$$[1x^{(0)}\,x^{(1)}\,\ldots\,x^{(n)}]X_i=[1xx^{(0)}\,\ldots\,x^{(i-1)}\,x^{(i+1)}\,\ldots x^{(n)}] \quad \text{(II)}$$

$$(i=0,1,2,\ldots\,n)$$

Multiply these in order by $x_1^{(0)}$, $x_1^{(1)}$, $\ldots x^{(n)}$, and add.

Thus—

$$[1x^{(0)}\,x^{(1)}\,\ldots\,x^{(n)}](x_1^{(0)}X_0+x_1^{(1)}X_1+\ldots+x_1^{(n)}X_n)$$

$$=x_1^{(0)}[1xx^{(1)}\,\ldots\,x^{(n)}]+\ldots+x_1^{(i)}[1x^{(0)}\,x^{(1)}\,\ldots\,x^{(i-1)}\,x^{(i+1)}\,\ldots x^{(n)}]$$

$$+\ldots\ldots=[1x^{(0)}\,x^{(1)}\,\ldots x^{(n)}]x_1$$

$$\therefore x_1=x_1^{(0)}X_0+x_1^{(1)}X_1+\ldots+x_1^{(n)}X_n,$$

and generally,

$$x_i=x_i^{(0)}X_0+x_i^{(1)}X_1+\ldots+x_i^{(n)}$$

$$(i=1,2,3,\ldots\,n).$$

Thus we see that all our results obtained in Cartesian system can be transformed into this new system by means of the above formulae of transformation. We do not propose to discuss the details of these transformations at present, but we shall only make use of the homogeneous nature of this new system of co-ordinates in studying certain properties of Hyper-surfaces, analogous to those in ordinary Geometry of three dimensions. From the above formulae it is easily seen that the co-ordinates of any point in an r-space contained in a higher space of n-dimensions can be expressed as a linear function of the co-ordinates of the generating points.

57. Hyper-surfaces :

A locus is generally defined to be an aggregate of a number (in general infinite) of points determined according to some specified law. Thus, if x is a point

$$a_0x_0+a_1x_1+\ldots\ldots+a_nx_n$$

in the space determined by $a_0, a_1, a_2, \ldots a_n$, then the equation $\phi(x_0, x_1, \ldots x_n)=0$, where ϕ is a homogeneous function, limits the arbitrary nature of the ratios $x_0 : x_1 : x_2 : x_n$. The corresponding values of x form a special aggregate of points out of all the points in the space.

A locus is sometimes defined by means of simultaneous equations such as $\phi_1=0, \phi_2=0, \ldots \phi_r=0$, where the ϕ's are all homegeneous, and a locus defined by r equations is said to be of $(n-r)$ dimensions. (These we have called "a Complex of order r" in the preceding chapter.) It is to be noticed that the functions $\phi_1, \phi_2, \ldots \phi_r$ are not necessarily linear.

We shall call a locus a "Hyper-surface" which is of one dimension less than the space containing it. Thus in plane Geometry a conic is a curve of one dimension, in the geometry of three dimensions a conicoid has a surface of two dimensions. Thus, in general, a Hyper-surface in an r-space is of $(r-1)$ dimensions.

A Hyper-surface defined by an equation of the rth degree will be called a surface-locus or a Hyper-surface of the rth order.

58. Consider the ∞^{n-1} aggregate of the points of an n-space, which satisfies a rational, integral and homogeneous equation of the rth order in the current co-ordinates : $x_0, x_1 x_2, \ldots x_n$.

$$f(x_0, x_1, x_2, \ldots x_n) = \sum a_{i_1} a_{i_2} \ldots a_{i_r} x_{i_1} x_{i_2} \ldots x_{i_r} \quad \ldots (1)$$

where the sum extends to all combinations of $i_1, i_2, \ldots i_r$ of the rth order, repetitions being allowed.

The aggregate of points defined by the equation (1) will be called a Hyper-surface of order r, and will be denoted by V^r_{n-1}.

The number of linear and homogeneous co-efficients in (1)

is $^{(n+1-r-1)}$[*] $C_r = {}^{n+r}C_r \equiv \left(\frac{n+r}{r} \right)$, say

Hence if we put $N(r) \equiv \left(\frac{n+r}{r} \right) - 1$, all the Hyper-surfaces of order r in an n-space form a "linear system $\infty^{N(r)}$"[†]

Thus the equation of a Hyper-surface of order r can be made to satisfy $N(r)$ conditions, and no more.

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[*] *Vide*—Algebra—Smith, §. 246.

[†] Bertini.

A surface is said to be "degenerate" or "non-degenerate" according as the function f breaks up or not into the product of two or more factors of lower orders.

A degenerate Hyper-surface is composed of two or more "non-degenerate" Hyper-surfaces.

59. The order r of a Hyper-surface has a geometric significance. It is the number of points in which a line intersects V^r_{n-1}. In fact, the equation (1) of § 58 and $(n-1)$ equations of a right line have r solutions common, which correspond to the roots of an equation of the rth degree in one co-ordinate (x_1 say), *i.e.* of the equation obtained by eliminating n of the $(n+1)$ co-ordinates between the above n equations. If any line has more than r points common with the Hyper-surface, it lies wholly on the surface, the equation having more than r roots reduces to an identity.

In general, if we have a k-omal,* to find its intersections with a V^r_{n-1}, we have the equation of the V^r_{n-1} and the $(n-k)$ equations of the omal. Thus we obtain altogether $(n-k+1)$ equations. By means of these we can eliminate $(n-k+1)$ of the $(n+1)$ co ordinates and the resulting equation involves only k co-ordinates. Thus the intersection is a V^r_{k-1}.

60. Intersections of Hyper-surfaces :

The aggregate of points common to two or more Hyper-surfaces is called their intersection. By a known theorem in Algebra, n hyper-surfaces $V^{r_1}_{n-1}$,$V^{r_2}_{n-1}$,...$V^{r_n}_{n-1}$ have in general $r_1 r_2 \ldots r_n$ points of intersection. If they have more than $r_1 r_2 \dots r_n$ points of intersection, they have necessarily an infinite number, forming one or more Complexes of different dimensions.

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* Cayley—Memoir on Abstract Geometry, Phil. Trans. of the Royal Soc. of London, Vol. CLX, 1870.

It follows that k hyper-surfaces ($k < n$) intersect $n-k$ hyper-planes (*i.e.* a space of k dimensions) in $r_1 r_2 r_3 \dots r_k$ points. This group of $r_1 r_2 r_3 \dots r_k$ points is called a Complex of order $r_1 r_2 \dots r_k$ and of dimension $n-k$, and is therefore denoted by $V^{r_1 r_2 \dots r_k}_{n-k}$. But if these Hyper-surfaces have a point and the Complex $V^{r_1 r_2 \dots r_k}_{n-k}$ common, any space S_k of k dimensions will intersect this Complex in $r_1 r_2 \dots r_k + 1$ points, and consequently in an infinite number of points.

61. If we consider $k+1$ hyper-surfaces of the same order r (linearly independent) and the linear system ∞^k determined by them,* all the Hyper-surfaces of the system have in common the points of the Complex of intersection (if it exists) of those Hyper-surfaces. This Complex is called the "base Complex" of the system, and if $k+1 \leq n$, it certainly exists and is a Complex $V^{r^{k+1}}_{n-k-1}$.

It can be proved very easily that the Hyper-surfaces of order r, which pass though $N(r)-k$ generating points of any n-space, constitute a linear system ∞^k, and, if $k \leq n-1$, they have in common a Complex $V^{r^{k+1}}_{n-k-1}$, determined by those $N(r)-k$ points. If $k = n-1$, they have r^n points common. In fact, if we substitute the coordinates of the $N(r)-k$ given points in the equation of a Hyper-surface, we can determine, in terms of $k+1$

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* *Vide*—Bertini, loc. cit. chap. X, §. 1.

co-efficients, all the remaining co-efficients as linear and homogeneous functions of them ; and these values substituted in the given equation will determine a system ∞^{k}.

62. Polarisation in n-dimensions :

Let $f(x_0, x_1, x_2, \ldots x_n) = 0$ be the equation of any Hyper-surface of order r. ... (1)

Let P and Q be the two points y_i and z_i $(i = 0, 1, 2, 3, \ldots n)$. Then the co-ordinates of any point on the line PQ may be taken as $\lambda y_i + \mu z_i$ $(i = 0, 1, 2, \ldots n)$. The co-ordinates of the r points of intersection of PQ with the Hyper-surface (1) are found by substituting these values for $x_0, x_1, \ldots x_n$ in (1), and then determining the ratio λ/μ from the resulting equation, which is of degree r.

The equation thus obtained is—

$$\lambda^r f_y + \binom{r}{1} \lambda^{r-1} \mu \Delta_z f_y + \binom{r}{2} \lambda^{r-2} \mu^2 \Delta_z^2 f_y + \ldots = 0 \quad (2)$$

where $\qquad f_y \equiv f(y_0, y_1, y_2, \ldots y_n)$

and $\qquad \Delta_z^s f_y \equiv \dfrac{1}{(r-s+1)\ldots r} \left(z_0 \dfrac{\partial f}{\partial y_0} + \ldots + z_n \dfrac{\partial f}{\partial y_n} \right)^s$

The operation Δ_z^s is called "*Polarisation*" with respect to the pole z.

The equation (2) can also be written in descending powers of μ and then comparing the coefficients with those of (2) we obtain the following identities :—

$$\Delta_z^s f_y \equiv \Delta_y^{r-s} f_z$$

$$\Delta_z^s \Delta_z^t f_y \equiv \Delta_z^{s+t} f_y$$

$$\Delta_z^s \Delta_z^t f_y \equiv \Delta_z^t \Delta_z^s f_y$$

The equation $\Delta_z^s f_y = \Delta_y^{r-s} f_z = 0$ represents a Hyper-surface of order $r-s$, which is called the sth polar of z with respect to V^r_{n-1}. In particular, the $(r-1)$th polar is a Hyper-plane and is called the Polar Hyper-plane of order $r-1$. The $(r-2)$th polar is called the "Polar Quadric,"* and so on.

Note : This is really an extension of the Theory of Poles and Polars in the Geometry of three dimensions, and consequently the theorems proved in this last case are also valid in the geometry of n dimensions. We give below some of those theorems without proof and shall try to interpret them geometrically on a future occasion.

63. **Theorem 1** : If the sth polar of P passes through Q, then the $(n-s)$th polar of Q passes through P. (Theorem of Reciprocity.)

Theorem 2 : The sth polar of a point P with respect to the kth polar of the same point with respect to a Hyper-surface V^r_{n-1} is the $(s+k)$th polar of P with respect to V^r_{n-1}.

Theorem 3 : The relation of a Hyper-surface to its polar Hyper-surfaces is a projective relation.

Theorem 4 : If we form the sth polar of a point P with respect to a Hyper-surface, and the kth polar of another point Q with respect to this polar Hyper-surface, and so on, we finally obtain a Hyper-surface which is independent of the order of Polarisation.

64. Consider that the point P(z) is any one (A_0) of the fundamental points *i.e.* the coordinates of P are (1,0,0, ... 0) and intersect the Hyper-surface V^r_{n-1}, given by the equation (1), by the fundamental Hyper-plane U_1, $x_1=0$ which passes through this point. The section will be a Hyper-surface of $x_1=0$, which will have its equation $[f]_{x_1=0}=0$, where $[f]_{x_1=0}$ represents the result of substituting $x_1=0$ in f.

---

* Consider the Hyper-surface formed by the faces of the fundamental Simplicissima. Its equation is therefore $x_0 x_1 x_2 \dots x_n = 0$.

Therefore the Polar Hyper-plane of y is $\sum y_0 y_1 \dots y_{i-1} x_i y_{i+1} \dots y_n = 0$ If the point y is the point (1,1,...1), the Polar hyper-plane is $\sum x_i = 0$ which is called the unit Hyper-plane.

The first polar of P with respect to this Hyper-surface will be

$$\frac{\partial [f]_{x_1=0}}{\partial x_0}$$

Now since

$$\frac{\partial [f]_{x_1=0}}{\partial x_0}=\left[\frac{\partial f}{\partial x_0}\right]_{x_1=0}$$

the above first polar is also the intersection of $x_1=0$ with the first polar of z with respect to V^r_{n-1} This property can be extended to the second polar of z with respect to V_r^{n-1}, since it is the first polar of the first polar.

Therefore any space S_{n-1} through a point z intersects a V^r_{n-1} and the sth polar of z with respect to V^r_{n-1} in a V^r_{n-2} and in the sth polar of z with respect to V^r_{n-2} (Hyper-surface of S_{n-1}).

In general, any linear space S_k, which passes through a point z, intersects a V^r_{n-1} and the sth polar of z with respect to V^r_{n-1} in a V^r_{k-1} and in the sth polar of z with respect to V^r_{k-1} (Hyper-surface of S_k). This is easily proved by applying the preceding theorem to the section of V^r_{n-1} by the Hyper-plane of S_k.

65. Tangent Spaces :

Next suppose that the point $P(y_i)$ lies on the Hyper-surface V^r_{n-1}. We find the condition that the line PQ meets the Hyper-surface in two coincident points. In this case we must have $f_y=0$ and $\Delta_z f_y=0$.

This equation, in which z is regarded as current coordinates, represents a Hyper-plane (passing through y). Hence the tangent lines to V^r_{n-1} at any of its generating points are contained in a Hyper-plane called the "Tangent Hyper-plane" to V^r_{n-1} at the point. This Hyper-plane is the polar Hyper-plane of y with respect to V^r_{n-1}. Conversely, if a point lies on its polar Hyper-plane, the point lies on the Hyper-surface.

66. Polar Quadrics :

The polar quadric of any point $P(y_i)$ with respect to any V^r_{n-1} has the equation of the form

$$\Delta^2_x f_y = \Sigma \frac{\partial^2 f_y}{\partial y_i \partial y_k} x_i x_k = 0.$$

This Quadric has a double point z_i *i.e.* is a cone, if

$$\Sigma \frac{\partial^2 f_y}{\partial y_i \partial y_k} z_i = 0, \ (k=0,1,2,\ldots n) \qquad \ldots \quad (1)$$

which shows that the first polar of z, given by

$$\Sigma \frac{\partial f_z}{\partial x_i} z_i = 0 \text{ has a double point at } y.$$

Hence if the polar quadric of P has a double point at Q, the first polar of Q has a double point at P, and vice versa.

The condition for the coexistence of the equations in (1) is—

$$\begin{vmatrix} \frac{\partial^2 f_y}{\partial y_0^2} & \frac{\partial^2 f_y}{\partial y_0 \partial y_1} & \cdots & \frac{\partial^2 f_y}{\partial y_0 \partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f_y}{\partial y_n \partial y_0} & \frac{\partial^2 f_y}{\partial y_n \partial y_1} & \cdots & \frac{\partial^2 f_y}{\partial y_n^2} \end{vmatrix} = 0.$$

and therefore the locus of the poles of the polar Quadrics with double points or the locus of the double points of the first polars is a Hyper-surface $V^{(n+1)(r-2)}_{n-1}$ of order $(n+1)(r-2)$, which is called the "Hessian."

Note: All these theorems are obtained as extensions of the theorems in ordinary Geometries. Many other similar theorems can be obtained in this way. But we postpone for the present the discussion of these theorems or other deductions. We consider the case of Quadric Hyper-surfaces, and in doing so we shall use the notation used by Whitehead in his Universal Algebra, Vol I.

67. Quadric Hyper-Surfaces :

Hyper-surfaces of the second order are called "Quadrics" or "Quadric Hyper-surfaces," and it is denoted by $V^2{}_{n-1}$.

The most general equation of a Quadric $V^2{}_{n-1}$ may be written as—

$$\Sigma a_0 x^2{}_0 + 2\Sigma a_{01} x_0 x_1 \equiv (a)(x) = 0, \text{ say.}$$

Let $$\Sigma a_{00} x_0 x'_0 + \Sigma a_{01}(x_0 x'_1 + x'_0 x_1) \equiv (a)(x)(x')$$

If z be a point on the line PQ, where P and Q are the points x_i, x'_i, its co-ordinates may be taken as

$$\lambda x_i + \mu x'_i \, (i = 0, 1, 2 \ldots n)$$

Substituting these values for the co-ordinates in the equation of the Quadric, we obtain

$$(a)(z)^2 = \lambda^2 (a)(x)^2 + 2\lambda\mu(a)(x)(x') + \mu^2 (a)(x')^2 = 0 \quad \ldots \quad (1)$$

This equation shows that if more than two points of a line lie on the Quadric, the whole line lies on it. If the equation (1) breaks up into two linear factors, the Quadric is degenerate and consists of two proper Hyper-planes of $(n-1)$ dimensions corresponding to the two factors.

An space of any dimension either intersects a Quadric in a Quadric Surface contained in that space *i.e.* in a Quadric Surface of dimensions lower by one than that of the space, or itself lies entirely in the Quadric.

68. Poles and Polars :

Consider the equation $(a)(x)(x')=0$. This equation may be regarded as defining the locus of the point x_i, when the other point x'_i remains fixed and vice versa. When x_i is regarded fixed, the locus is the polar of x'_i with respect to the Quadric $(a)(x)^2=0$. The polar of a point will of course be a Hyper-plane of $(n-1)$ dimensions. The point x'_i is called the "Pole."

The ordinary theorems on Poles and Polars obviously hold :

(*i*) If any point x'_i lies in its polar $(a)(x)(x')=0$, we must have $(a)(x')(x')=0$, *i.e.* $(a)(x')^2=0$. Hence x'_i lies on the Quadric itself. Thus all the points on the Quadric are reciprocally polars to themselves, or they may be called "Self-polar" with respect to the Quadric. In this case the polar $(a)(x)(x')=0$ is the "tangent-space"* at the point x'_i.

(*ii*) Again, if the point lies in an $(n-1)$-space, the polar spaces all pass through a common point which is the pole of the $(n-1)$-space with respect to the Quadric.

If the point lies in an $(n-2)$-space given as the intersection of two $(n-1)$-spaces, the polar-spaces all pass through the two poles of the defining $(n-1)$-spaces and have therefore a common line of intersection.

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* "Berührungs-Raum"—Veronese.

And generally, if the point lies in an r-space, the polar spaces of the points must pass through the $(n-r)$ poles of the defining $(n-1)$-spaces and thus have an $(n-r-1)$-space of intersection.

Thus we see that spaces can be connected by a reciprocal relation *i.e.* that of poles and polars of the above nature :

An S_{n-1} space has one pole defining an space of zero dimension,
,, S_{n-2} ,, ,, two poles ,, ,, one dimension,
,, S_{n-3} ,, ,, three ,, ,, ,, two dimensions,
,, S_{n-4} ,, ,, four ,, ,, ,, three ,,
... ...
,, S_{n-r} ,, ,, r ,, ,, ,, $(r-1)$,,
... ...
etc. etc. etc.

From these we infer that two spaces can generally be connected by a dual relation such that the sum of their dimensions is always equal to $(n-1)$.*

We shall define two spaces S_r and S_k $(r+k=n-1)$ as "*Conjugate spaces,*" when there is a dual relation between them of the above nature.

Thus the two conjugate spaces are such that the polars of all points in one with respect to a given Quadric have the other common between them.

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* Veronese—" Ebenso heissen dual oder correlatif die Raüme, von denen der eine durch eine gewisse Anzahl unabhängiger Punkte und der andre durch eine gleiche Anzahl unabhangiger Raüme bestimmt wird."

Veronese has sought to establish a dual relation between spaces by the number of points and the number of equations determining them. Thus, n equations determine a point of zero dimension and n points determine an $(n-1)$-space ; $(n-1)$ equations determine a line of one dimension and $(n-1)$ points determine an $(n-2)$-space, and so on. Veronese—Grundzüge der etc., §. 159, p. 571.

69. Harmonic Properties :

Consider the points in which the line joining two conjugate points $P(x_i)$ and $Q(x_i')$ intersects the Quadric. Any point on the line has coordinates of the form $\lambda x_i+\mu x'_i, (i=0,1,2,\dots n)$.

Then the equation (2) of §. 62 becomes :—

$$\lambda^2(a)(x)^2+\mu^2(a)(x')^2=0 \qquad \dots \quad (1)$$

since $$(a)(x)(x')=0.$$

The two points of intersection must therefore be of the form $\lambda x_i \pm \mu x'_i$. Therefore the two conjugate points x_i and x'_i and these two points of intersection of the line PQ with the Quadric V^2_{n-1} form a Harmonic Range.*

70 We have seen that $(a)(x)(x')=0$ is the tangent space to the Quadric at the point x'_i.

Let $\alpha_0x_0+\alpha_1x_1+\dots+\alpha_nx_n=0$ be the equation of an $(n-1)$-space. The condition that this should be a tangent-space to V^2_{n-1} is obviously—

$$\begin{vmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0n} & \alpha_0 \\ a_{10} & a_{11} & a_{12} & \dots & a_{1n} & \alpha_1 \\ a_{20} & a_{21} & a_{22} & \dots & a_{2n} & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & a_{n2} & \dots & a_{nn} & \alpha_n \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_n & 0 \end{vmatrix}=0.$$

This condition can be written in the form

$$\Sigma A_{00}\alpha^2_0+2\Sigma A_{01}\alpha_0\alpha_1=0$$

or $$(A)(\alpha)^2=0 \qquad \dots \quad (A)$$

where $A_{00}=\dfrac{\delta\Delta}{\delta a_{00}}$, etc. Δ being the determinant

$$(a_{00}\ a_{11}\ a_{22}\dots a_{nn}).$$

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* C. A. Scott : Modern Analytical Geometry, §. 40.

If now we denote the coordinates of the given $(n-1)$-space by λ_i $(i=0,1,2,\dots n)$, then the condition that this may be a tangent-space can be written as

$$(A)(\lambda)^2=0 \qquad \dots \quad (B)$$

Thus we have two conditions :—(1) the point x_i lies on the surface $(a)(x)^2=0$, and, (2) the space λ_i is a tangent-space to the surface $(A)(\lambda)^2=0$.

The equation (B) may be called the tangential equation of the Quadric V^2_{n-1}.

Thus a Quadric may be generated in two ways : (1) as the locus of a point ; (2) as the envelope of a tangent-space.*

71. Self-polar Simplicissima :

We can choose the fundamental Simplicissima in such a manner that the fundamental points are reciprocally polar to each other with respect to a given Quadric.

Let $(a)(x)^2=0$ be the equation of a Quadric.

The coordinates of the fundamental point A_0 are $(1,0,0,0,\dots)$. Its polar space is to be the fundamental $(n-1)$-space U_0. But the equation of its polar space is $(a)(x)(1)=0$.

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[*] *Cf.* Prof. Cayley—* * * Thus we arrive at the notion of double generation of a k-fold locus—such locus is the locus of points, or say, of the "*ineunt*" points thereof ; and it is also the envelope of the tangent-omals thereof. We have thus a theory of duality. This theory is essential to the systematic development of n-dimensional Geometry, the original classification of loci as 2-fold, 3-fold etc. is incomplete and must be supplemented with the loci reciprocally connected with these loci respectively. And moreover the theory of the singularity of a locus can only be systematically established by means of the same theory of duality. * *

Phil. Trans. of R. Soc. of London, Vol. CLX, 1870.

∴ We must have $a_{01}=a_{02}=\dots=a_{0n}=0$. Similarly in other cases.

Thus the equation of the Quadric is reduced to the form—

$$a_{00}x^2_0+a_{11}x^2_1+\dots+a_{nn}x^2_n=0 \qquad \dots \quad \text{(C)}$$

we call this the "self-polar Simplicissima."

In order to construct a self-polar Simplicissima with respect to a given Quadric we proceed as follows :—

Let A_0 be any point not on the Quadric. Take the polar space of A_0 with respect to V^2_{n-1}, which is of $(n-1)$ dimensions and does not contain A_0. The intersection of this with V^2_{n-2} is another Quadric of $(n-2)$ dimensions* V^2_{n-2} contained in it. Take any point A_1 in this polar-space not on the Quadric. Next take any point A_2 on the intersection of the polar-spaces of A_0 and A_1, and so on. Thus ultimately $(n+1)$ independent points are obtained, all reciprocally polar to each other with respect to a given Quadric.

72. Generating Spaces of a Quadric :

In ordinary Geometry of three dimensions we have the proposition : "Through any point of a conicoid generating lines, real or imaginary, can be drawn." This proposition has been generalised by Veronese† and stated in the form :—

"If the complete space be of 2μ or $2\mu-1$ dimensions, the subordinate spaces, real or imaginary, contained within any Quadric surface will be of $(\mu-1)$ dimensions."

We have seen in the preceding article that $(n+1)$ points of space can be determined reciprocally polar to each other.

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* *Vide*—Bertini, loc. cit. Chap. VIII, §§. 2-3.

† *Vide*—Veronese, loc. cit.

Now the question is to determine the number of such points which can be found to lie on the Quadric surface.

Now, if α_1 be a point on the Quadric, it lies in the polar space $(a)(x)(\alpha_1)=0$. If α_2 be another point on the Quadric lying in this polar, $(a)(\alpha_1)(\alpha_2)=0$, and the points α_1, α_2 are such that each lies in the polar of the other. Hence any point on the line $\alpha_1\alpha_2$ lies in both polars and on the Quadric. But the polars intersect in an $(n-2)$-space.

Now take another point α_3 on the intersection of this $(n-2)$-space with the Quadric. Then $\alpha_1, \alpha_2, \alpha_3$ are reciprocally polar to each other and any point in the plane $(\alpha_1, \alpha_2, \alpha_3)$ lies in the intersection of the three polar spaces and on the Quadric. But the intersection of three polar-spaces is an space of $(n-3)$ dimensions. Next take another point α_4 on the intersection of this $(n-3)$-space with the Quadric, and so on. Suppose that we have thus selected k points reciprocally polar to each other, which are of course such that each lies on the polar-spaces of others, as well as on that of itself (since it lies on the Quadric). If these k points be independent, their polars intersect in an space of $(n-k)$ dimensions,* which contains the k points. Since these k points are independent, we must have

$$n-k+1\geq k$$

$\therefore\ 2k\leq n+1$. Thus the greatest value of k is the highest number of integers in $\frac{n+1}{2}$. Thus if $n=2\mu$ or $n=2\mu-1$, the value of k is μ.

Therefore we have k or μ independent points lying on the Quadric and these define an space of $(\mu-1)$ dimensions contained in the surface of the Quadric.

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[*] See §. 68.

73. Quadric Hyper-cone :

Corresponding to the ordinary Cone in the Geometry of three dimensions, we have a Quadric Hyper-surface in n-space which will be called a "Conical Quadric Hyper-surface."

Veronese has given the following definition[*] of the general Conical Quadric Hyper-surface :—

The figure formed by the lines which join the point V_0 with the points of the system $F^m{}_p$ is called "Kegelfläche" or "Kegel of the first kind." ($F^m{}_p$ is a Hyper-surface of order m and dimension p. V_0 is the vertex, and $F^m{}_p$ is the base system of the cone. The lines which join V_0 with the points of the base are called the generators of the Hyper-cone.

To find the condition that the general equation of the second degree should represent a cone.[†]

The equation is written as $(a)(x)^2=0$ ... (I)

Let (a) be the vertex V_0 and x'_i any point on the Cone. Then for all values of λ and μ, the point whose coordinates are $\lambda x'_i+\mu a_i$ must lie on the surface.

Hence substituting in the equation (1) of the surface we obtain—

$$\lambda^2(a)(x')^2+2\lambda\mu(a)(a)(x')+\mu^2(a)(a)^2=0 \quad \text{... (II)}$$

But (a) and (x') lie on the surface.

$$\therefore\ (a)(a)^2=0 \text{ and } (a)(x')^2=0.$$

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[*] "Die Figur, welche von allen Graden gebildet wird die den Punkt V_0 mit den Punkten des Systems $F^m{}_p$ verbinden, heisst Kegelfläche oder Kegel erster Art. V_0 ist die "Spitze" und $F^m{}_p$ des leitende System des Kegels. Die Graden, welche die Spitze mit den Punkten des leitenden Systems verbinden, heissen die "Erzeugeden" des Kegels. Veronese—loc. cit. §. 179, Def^n I.

[†] For brevity of expression we write "Cone" for "Conical Quadric Hyper-surface."

$$\therefore\ (a)(\alpha)(x')=0 \text{ for all positions of } x'.$$

Now, $(a)(\alpha)(x')=0$ can be written as—

$$a_{00}\alpha_0 x'_0+a_{11}\alpha_1 x'_1+\ \dots\ +a_{01}(\alpha_0 x'_1+\alpha_1 x'_0)+\ \dots\ =0,$$

or $\{a_{00}\alpha_0+a_{01}\alpha_1+\ \dots\ +a_{0n}\alpha_n\}x'_0+\{a_{10}\alpha_0+a_{11}\alpha_2+$

$$\dots\ +a_{1n}\alpha_n\}x'_1+\ \dots\ +\ \dots\ =0 \quad \dots \quad \text{(III)}$$

This is to be satisfied for all positions of the point (x') *i.e.* the coefficients of $x'_0, x'_1, \dots x'_n$ in (III) must identically vanish.

Eliminating the $(n+1)$ quantities $\alpha_0, \alpha_1, \dots \alpha_n$ between the $(n+1)$ equations thus obtained we obtain the condition $\Delta=0$, where Δ is the determinant

$$\Delta\equiv\begin{vmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \\ a_{20} & a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Hence the condition that the general equation of the second degree should represent a conical Hyper-surface is the vanishing of the determinant Δ.

74. Consider the two Quadric Hyper-surfaces

$$(a)(x)^2\equiv S=0 \text{ and } (a')(x)^2\equiv S'=0.$$

Then $S+\lambda S'=0$ represents a quadric Hyper-surface which passes through the intersections of S and S'.

This will represent a conical Quadric Hyper-surface, if the determinant $(a_{00}+\lambda a'_{00}, a_{11}+\lambda a'_{11}, \dots a_{nn}+\lambda a'_{nn})=0$.

This is an equation of the $(n+1)$th degree in λ and therefore there are $(n+1)$ values of λ for which $S+\lambda S'=0$ may represent a conical Hyper-surface.

Thus in general $(n+1)$ conical Quadric Hyper-surfaces can be drawn through the intersections of two Quadric Hyper-surfaces.

We may prove also that " the vertices of these conical Quadric Hyper-surfaces form a Self-polar Simplicissima." This is really an extension of a theorem in three dimensional Geometry.

75. Sections of a Conical Hyper-surface :

" The Conical Hyper-surface $V_0 - F^m{}_p$* of p dimensions and order m is intersected by an space S_k of k dimensions through the vertex V_0, in general, in a Conical Hyper-surface $V_0 - F^m{}_{k+p-n}$ of $k+p-n+1$ dimensions and order m which can reduce to a system of m lines."†

Any space S_{n-p} drawn through the vertex V_0 intersects the base-space S_{n-1} in an space S_{n-p-1} which has at the most m points common with $F^m{}_p$. These points with the vertex form the m generators of the Conical Hyper-surface which lie in the space S_{n-p}. Each space S_{k-1} contained in S_{n-1} intersects $F^m{}_p$ at the most in a surface F_{k+p-n} of $k+p-n$ dimensions.‡ If we join the vertex V_0 with S_{k-1}, we obtain an space S_k which intersects the conical Hyper-surface in a Conical-surface $V_0 - F^m{}_{k+p-n}$ of $k+p-n+1$ dimensions and order m.

If $k=n-1$, then the conical Hyper-surface of section is of p dimensions. If $p=n-2$, then a plane through the vertex intersects $V_0 - F^m{}_{p-1}$ at the most in m lines.

Corollary :

If the Hyper-surface $F^m{}_p$ in S_{n-1} is a Hyper-surface of the second order, or a spheric $S^2{}_{n-2}$, then the conical Hyper-surface is of the second order.

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* This notation is used by Veronese—loc. cit. §. 179. Def^n I.

† Cf. Veronese—loc. cit. §. 179, Satz I.

‡ Veronese—loc. cit. §. 178, Satz I.

In this case each plane through the vertex cannot intersect it in more than two generators, and in general, an space of r dimensions through the vertex intersects it in a conical hyper-surface of the second order of $(r-1)$ dimensions $(r=3,4, \ldots n-1)$.

76. Veronese has given another form of a conical hyper-surface : We give here the definition of such a surface :—

" We shall consider two dual spaces S_m and S_{n-m-1} which do not intersect each other."

Let $F^r{}_p$ be a given hyper-surface in S_{n-m-1} and let $p<n-m-1$.

The space of $m+1$ dimensions, which joins the space S_m with the points of the hyper-surface $F^r{}_p$, forms a conical hyper-surface or a cone of the rth order, which is of $p+m+1$ dimensions ; S_m is the vertex-space, $F^r{}_p$ is the base-surface of the cone. The spaces which join S_m with the points of the base-surface are called " generating spaces."

CHAPTER V.

PARAMETRIC REPRESENTATION.

77. The literature on the Geometry of Hyper-surfaces is very limited and most of the few authors who have seen their way to deal with the subject have taken up a standpoint, by no means based on elementary ideas, but to a high degree technical, depending on intricate differential formulae. Having but imperfect knowledge of continental languages the writer is not in a position to fully ascertain how far the authors in those languages have adopted familiar elementary ideas in developing the Geometry of Hyper-surfaces with the application of differential methods, but in almost all treatises on differential Geometry references are made as to the suitability of these methods to higher Geometries. The present writer only avails himself of those references and ventures, though in some cases lacking confidence in their generality and Geometrical interpretation, to generalise the formulae used in the ordinary differential geometries, not knowing how far he has been anticipated by others. Kommerell[*] in his dissertation has given a treatment of two-dimensional spreads in four-dimensions. The present paper will deal with surfaces in a four-space.

A single relation in four variables (x,y,z,t) represents a Hyper-surface in four-space. Along the hyper-surface, each of the variables can be conceived as expressible in terms of three variable parameters. Two relations in four variables represent a surface (two-dimensional spread) in four-space. Along this surface each of the variables can be considered as expressible in terms of two parameters. In a similar manner, three relations in four variables may define a curve in four-space along which each of the variables can be considered as expressible in terms of a single parameter. Four such relations, in general, define a point or a number of points. Hence it follows that two hyper-surfaces in four-space intersect in a surface or surfaces, a hyper-surface and a surface intersect in a curve or curves and two surfaces intersect in a point or points.

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[*] Karl Kommerell—Riemannsche Flächen im ebenen Raum von vier Dimensionen—Math. Ann. Bd. 60.

Starting with these preliminary notions we can assert that the equations—

$$x=f_1(u,v,w),\ y=f_2(u,v,w),\ z=f_3(u,v,w),\ t=f_4(u,v,w)\ \ldots \quad (1)$$

where u,v,w are variable parameters, define a hyper-surface.

Assigning particular values to any one of the parameters (say u) the locus of the point (x,y,z,t), as v and w vary, is a surface on the hyper-surface, for (x,y,z,t) are now functions of two parameters. Suppose now that the three parameters are connected by the relation

$$\phi(u,v,w)=0 \quad \ldots(2)$$

Equations (1) and (2) will then define a two-dimensional spread or a surface on the hyper-surface.

In particular, any one of the three relations $u=a$(const.) $v=b$ (a const.), $w=c$ (a const.) may be taken corresponding to equation (2) and in that case each of them may be taken to represent a surface. These may be called "parametric surfaces" on the hyper-surface. The three surfaces meet in a point P, whose position may then be considered as determined by the values of the parameters u, v and w and these values may then be called the "Curvillinear Co-ordinates" of P.

The functions A,B,C, ... thus defined are analogous to the functions F,G,H used by Gauss.* For any other surface, the expression (5) will have the same form, subject to a different relation between du, dv, dw depending upon the particular surface selected. Consequently the value of ds given by (5) is the element of arc on any surface of the hyper-surface. This may be called the linear element of the hyper-surface.

78. The surfaces $v=b$, $w=c$ intersect in a curve OL on the hyper-surface. Similarly, $w=c$, $u=a$ intersect in OM and $u=a$, $v=b$ intersect in ON. The curves OL, OM, ON may be called the "parametric curves" on the hyper-surface.

The direction-cosines of the tangent lines to these curves are respectively proportional to—

$$x_u, y_u, z_u, t_u \;:\; x_v, y_v, z_v, t_v \;:\; x_w, y_w, z_w, t_w \;:$$

where the suffixes indicate differentiation with respect to the parameters.

Now, if l_1, l_2, l_3, l_4 be the direction-cosines of the normal at O to the hyper-surface, since it is perpendicular to the three tangent lines, the following relations hold :—

$$\left.\begin{aligned} l_1 x_u + l_2 y_u + l_3 z_u + l_4 t_u &= 0 \\ l_1 x_v + l_2 y_v + l_3 z_v + l_4 t_v &= 0 \\ l_1 x_w + l_2 y_w + l_3 z_w + l_4 t_w &= 0 \end{aligned}\right\}$$

Solving these equations for l_1, l_2, l_3, l_4, we obtain

$$\frac{l_1}{\alpha} = \frac{l_2}{\beta} = \frac{l_3}{\gamma} = \frac{l_4}{\delta} = \frac{1}{\sqrt{\Sigma\alpha^2}} \qquad \ldots \quad (6)$$

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* Disquisitiones generales cira superficies curvas (Eng. Trans. by Morehead and Hiltebite).

where

$$\alpha \equiv \frac{\partial(yzt)}{\partial(u,v,w)},\ \beta \equiv \frac{\partial(tzx)}{\partial(u,v,w)},\ \gamma \equiv \frac{\partial(xyt)}{\partial(u,v,w)},\ \delta \equiv \frac{\partial(xyz)}{\partial(u,v,w)}.$$

On simplification it will be found that

$$\alpha^2+\beta^2+\gamma^2+\delta^2=\text{ABC}+2\text{FGH}-\text{AF}^2-\text{BG}^2-\text{CH}^2$$

$$=\triangle \text{ (say)}$$

$$\therefore\ l_1/\alpha=l_2/\beta=l_3/\gamma=l_4/\delta=\frac{1}{\sqrt{\triangle}}.$$

Thus the direction-cosines of the normal at O are

$$\frac{\alpha}{\sqrt{\triangle}},\ \frac{\beta}{\sqrt{\triangle}},\ \frac{\gamma}{\sqrt{\triangle}},\ \frac{\delta}{\sqrt{\triangle}}.$$

If Θ be the hyper-plane angle between the tangents to the parametric curves, we must have—

$$\text{ABC}\sin^2\Theta=\sum\begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}^2$$

$$=\text{ABC}+2\text{FGH}-\text{AF}^2-\text{BG}^2-\text{CH}^2=\triangle$$

$$\therefore\ \sin^2\Theta=\frac{\triangle}{\text{ABC}}. \qquad \ldots (7)$$

The direction-cosines of the tangent plane to the parametric surface $u=a$, which is determined by the tangent lines to the parametric curves OM, ON, are given by

$$\left\Vert \begin{matrix} x_v & y_v & z_v & t_v \\ x_w & y_w & z_w & t_w \end{matrix} \right\Vert$$

If $a_1, b_1, c_1, f_1, g_1, h_1$ are proportional to the direction-cosines.

$$a_1 = \frac{\partial(yz)}{\partial(v,w)}, \quad b_1 = \frac{\partial(zx)}{\partial(v,w)}, \quad c_1 = \frac{\partial(xy)}{\partial(v,w)},$$

$$f_1 = \frac{\partial(xt)}{\partial(v,w)}, \quad g_1 = \frac{\partial(yt)}{\partial(v,w)}, \quad h_1 = \frac{\partial(zt)}{\partial(v,w)}.$$

But $a^2_1 + b^2_1 + c^2_1 + f^2_1 + g^2_1 + h^2_1$

$$= \sum(x_v y_w - x_w y_v)^2 = \sum x^2_v \cdot \sum x^2_w - (\sum x_v x_w)^2$$

$$= \mathrm{BC} - \mathrm{F}^2$$

∴ The direction-cosines of the tangent plane to the surface $u=a$ are—

$$a_1/\sqrt{\mathrm{BC}-\mathrm{F}^2}, \quad b_1/\sqrt{\mathrm{BC}-\mathrm{F}^2}, \quad c_1/\sqrt{\mathrm{BC}-\mathrm{F}^2}$$

$$f_1/\sqrt{\mathrm{BC}-\mathrm{F}^2}, \text{ etc.} \qquad \ldots \quad (8)$$

Similarly, the direction-cosines of the tangent planes to the surfaces $v=b$ and $w=c$ are respectively

$$a_2/\sqrt{\mathrm{CA}-\mathrm{G}^2}, \quad b_2/\sqrt{\mathrm{CA}-\mathrm{G}^2}, \text{ etc.} \qquad \ldots \quad (9)$$

and $$a_3/\sqrt{\mathrm{AB}-\mathrm{H}^2}, \quad b_3/\sqrt{\mathrm{AB}-\mathrm{H}^2}, \text{ etc.} \qquad \ldots \quad (10)$$

where $$a_2 = \frac{\partial(yz)}{\partial(u,w)}, \text{ etc. and } a_3 = \frac{\partial(yz)}{\partial(u,v)}, \text{ etc.}$$

If θ_1 be the angle between the tangent lines to OM, ON, we have—

$$\cos\theta_1 = \frac{x_v x_w + y_v y_w + z_v z_w + t_v t_w}{\sqrt{\sum x^2_v} \cdot \sqrt{\sum x^2_w}} = \frac{\sum x_v x_w}{\sqrt{\mathrm{BC}}} = \frac{\mathrm{F}}{\sqrt{\mathrm{BC}}}. \qquad (11)$$

Similarly, if θ_2, θ_3 be the angles between the tangent lines ON, OL and OL, OM respectively, we have—

$$\cos\theta_2 = \frac{\mathrm{G}}{\sqrt{\mathrm{CA}}}, \text{ and } \cos\theta_3 = \frac{\mathrm{H}}{\sqrt{\mathrm{AB}}} \qquad \ldots \quad (12)$$

$$\therefore \quad \sin^2\theta_1 = 1 - \frac{F^2}{BC} = \frac{BC - F^2}{BC}$$

$$\therefore \sin\theta_1 = \sqrt{\frac{BC - F^2}{BC}}\ ; \text{similarly, } \sin\theta_2 = \sqrt{\frac{CA - G^2}{CA}}$$

and
$$\sin\theta_3 = \sqrt{\frac{AB - H^2}{AB}} \qquad \dots \quad (13)$$

If Θ_1 be the angle between the planes (9) and (10), we have

$$\sin\theta_2 \sin\theta_3 \cos\phi_1{}^{*} = \frac{a_2 a_3 + b_2 b_3 + c_2 c_3 + f_2 f_3 + g_2 g_3 + h_2 h_3}{\sqrt{CA - G^2} \cdot \sqrt{AB - H^2}}$$

$$\therefore \quad \cos\phi_1 = \frac{A\sqrt{BC}\ (a_2 a_3 + b_2 b_3 + \dots)}{(CA - G^2)(AB - H^2)}$$

$$= \frac{A\sqrt{BC}\ (GH - AF)}{(CA - G^2)(AB - H^2)} \qquad \dots \quad (14)$$

Similarly,
$$\cos\phi_2 = \frac{B\sqrt{AC}\ (FH - BG)}{(BC - F^2)(AB - H^2)} \qquad \dots \quad (15)$$

$$\cos\phi_3 = \frac{C\sqrt{AB}(FG - CH)}{(BC - F^2)(AC - G^2)} \qquad \dots \quad (16)$$

Definition : When three families of surfaces upon a hyper-surface are such that through any point, one and only one surface of each family passes, and the tangent planes at a point to the three surfaces through it are mutually perpendicular, the surfaces may be said to form a "*triply orthogonal system*" of surfaces.

Now, if the three tangent planes are mutually orthogonal, we have—

$$\cos\phi_1 = \cos\phi_2 = \cos\phi_3 = 0$$

$$\therefore \ GH - AF = FH - BG = FG - CH = 0$$

i.e.
$$FGH = AF^2 = BG^2 = CH^2$$

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[*] See §. 1.

Hence, a necessary condition that the parametric surfaces upon a hyper-surface form an orthogonal system is that

$$\mathrm{FGH}=\mathrm{AF}^2=\mathrm{BG}^2=\mathrm{CH}^2 \quad \ldots \quad (17)$$

This condition is also sufficient.

We have seen that the six direction-cosines of a plane are connected by the relations*

$$\left.\begin{array}{r} a^2+b^2+c^2+f^2+g^2+h^2=1 \\ af+bg+ch=0 \end{array}\right\}$$

It is easily seen that the direction-consines of the three tangent planes satisfy similar relations.

79. The quantities A,B,C, ... as defined above have important applications in the differential Geometry of Hyper-surfaces and are independent of the particular selection of orthogonal co-ordinate axes.

Consider the effect of orthogonal transformation :—

$$x'=a+l_1x+l_2y+l_3z+l_4t$$

$$y'=b+m_1x+m_2y+m_3z+m_4t$$

$$z'=c+n_1x+n_2y+n_3z+n_4t$$

$$t'=d+p_1x+p_2y+p_3z+p_4t$$

We have then

$$\begin{aligned} \mathrm{A}'=\sum x'^2{}_u &=\sum(l_1x_u+l_2y_u+l_3z_u+l_4t_u)^2 \\ &=x^2{}_u\sum l^2{}_1+y^2{}_u\sum l^2{}_2+z^2{}_u\sum l^2{}_3+t^2{}_u\sum l^2{}_4 \\ &\quad +2x_uy_u\sum l_1l_2+2z_ux_u\sum l_3l_1+2x_ut_u\sum l_1l_4 \\ &\quad +2y_uz_u\sum l_2l_3+2y_ut_u\sum l_2l_4+2z_ut_u\sum l_3l_4. \\ &=x^2{}_u+y^2{}_u+z^2{}_u+t^2{}_u \\ &=\mathrm{A}\left[\because \sum l^2{}_1=1 \text{ etc. and } \sum l_1l_2=\sum l_1l_3=\ldots=0\right] \end{aligned}$$

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[*] See §. 13.

Similarly,

$$F' = \sum x'_v x'_w = \sum (l_1 x_v + l_2 y_v + l_3 z_v + l_4 t_v)(l_1 x_w + l_2 y_w + l_3 z_w + l_4 t_w)$$

$$= x_v x_w + y_v y_w + z_v z_w + t_v t_w$$

$$= F$$

The quantities A,B,C,F,G,H may be called the "fundamental magnitudes" of the first order.

80. The differential form of equation of a hyper-surface can be very easily obtained, by considering the fact that any normal to the hyper-surface is perpendicular to any direction in the tangential space. Let dx, dy, dz, dt denote any direction in the tangential space and P,Q,R,S the direction-consines of the normal at a point O.

Evidently, therefore, we have

$$P\,dx + Q\,dy + R\,dz + S\,dt = 0 \qquad \dots \quad (19)$$

which is the differential equation of the hyper-surface.

When x, y, z, t are given as functions of u, v, w and consequently P,Q,R,S are deduced, the equation is satisfied identically. This is as it should be, for the integral equation is implicitly contained in the expressions for x, y, z, t.

If P,Q,R,S be given as appropriate functions of x, y, z, t, the equation $P\,dx + Q\,dy + R\,dz + S\,dt = 0$ represents a hyper-surface $\phi = \text{const.}$, only when

$$P:Q:R:S = \frac{\partial \phi}{\partial x} : \frac{\partial \phi}{\partial y} : \frac{\partial \phi}{\partial z} : \frac{\partial \phi}{\partial t}. \qquad \dots \quad (20)$$

Hence, P,Q,R,S must satisfy a certain relation, which we proceed to determine.

From equations (20) we have

$$\frac{\partial \phi}{\partial x} = \mu P, \quad \frac{\partial \phi}{\partial y} = \mu Q, \quad \frac{\partial \phi}{\partial z} = \mu R, \quad \frac{\partial \phi}{\partial t} = \mu S,$$

where μ is indeterminate.

$$\therefore \frac{\partial}{\partial y}(\mu \mathrm{P}) = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}(\mu \mathrm{Q})$$

or
$$\mu \frac{\partial \mathrm{P}}{\partial y} + \mathrm{P}\frac{\partial \mu}{\partial y} = \mu \frac{\partial \mathrm{Q}}{\partial x} + \mathrm{Q}\frac{\partial \mu}{\partial x}$$

or
$$\mu\left(\frac{\partial \mathrm{P}}{\partial y} - \frac{\partial \mathrm{Q}}{\partial x}\right) = \mathrm{Q}\frac{\partial \mu}{\partial x} - \mathrm{P}\frac{\partial \mu}{\partial y} \quad \ldots \quad (21)$$

Similarly,
$$\mu\left(\frac{\partial \mathrm{Q}}{\partial z} - \frac{\partial \mathrm{R}}{\partial y}\right) = \mathrm{R}\frac{\partial \mu}{\partial y} - \mathrm{Q}\frac{\partial \mu}{\partial z} \quad \ldots \quad (22)$$

$$\mu\left(\frac{\partial \mathrm{R}}{\partial x} - \frac{\partial \mathrm{P}}{\partial z}\right) = \mathrm{P}\frac{\partial \mu}{\partial z} - \mathrm{R}\frac{\partial \mu}{\partial x} \quad \ldots \quad (23)$$

$$\mu\left(\frac{\partial \mathrm{R}}{\partial t} - \frac{\partial \mathrm{S}}{\partial z}\right) = \mathrm{S}\frac{\partial \mu}{\partial z} - \mathrm{R}\frac{\partial \mu}{\partial t} \quad \ldots \quad (24)$$

$$\mu\left(\frac{\partial \mathrm{P}}{\partial t} - \frac{\partial \mathrm{S}}{\partial x}\right) = \mathrm{S}\frac{\partial \mu}{\partial x} - \mathrm{P}\frac{\partial \mu}{\partial t} \quad \ldots \quad (25)$$

$$\mu\left(\frac{\partial \mathrm{Q}}{\partial t} - \frac{\partial \mathrm{S}}{\partial y}\right) = \mathrm{S}\frac{\partial \mu}{\partial y} - \mathrm{Q}\frac{\partial \mu}{\partial t} \quad \ldots \quad (26)$$

If now we multiply (21) by (24), (22) by (25), (23) by (26) and add, we obtain

$$\left(\frac{\partial \mathrm{P}}{\partial y} - \frac{\partial \mathrm{Q}}{\partial x}\right)\left(\frac{\partial \mathrm{R}}{\partial t} - \frac{\partial \mathrm{S}}{\partial z}\right)$$
$$+\left(\frac{\partial \mathrm{Q}}{\partial z} - \frac{\partial \mathrm{R}}{\partial y}\right)\left(\frac{\partial \mathrm{P}}{\partial t} - \frac{\partial \mathrm{S}}{\partial x}\right)$$
$$+\left(\frac{\partial \mathrm{R}}{\partial x} - \frac{\partial \mathrm{P}}{\partial z}\right)\left(\frac{\partial \mathrm{Q}}{\partial t} - \frac{\partial \mathrm{S}}{\partial y}\right) = 0,$$

which is the condition of integrability to be satisfied. Hence the direction-cosines of the normal to a hyper-surface must satisfy this condition.

The converse theorem is also true.